Substructural Content

by

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1.0 Introduction

The goal of this dissertation is to put forward a theory of content and see how far it can take us. The theory of content amounts to the suggestion that we understand the content of a sentence in terms of its contribution to good implication. The twist is that I also acknowledge that good implications may be radically substructural, where "structural" refers to structural proof rules in sequent calculi (e.g. such rules as monotonicity, contraction, transitivity, and reflexivity).[1] Substructural then means capable of violating these rules. I therefore call such content *substructural content*, and I suggest that this theory can take us quite far.

1.1 An Outline

The dissertation is divided into three parts. In the first part (Ch. 2), I articulate my theory of meaning and explain the relevant background. The main goal of this chapter, aside from putting the theory on the table, is to convince the reader that the formal semantics I construct alongside the theory is completely tractable. I do this by showing that we can introduce the standard logic connectives and that simple and familiar proof rules (in the form of a sequent calculus) are sound and complete for the semantics. I also prove some results for the proof system that will be of particular interest to philosophers and logicians interested in inferentialism. Using the proof system, I give a precisification of logical expressivism. Logical

[1] See Figure 2 (p. 11, reprinted in this footnote):

Monotonicity

$$\frac{\Gamma \vdash \Theta}{\Delta, \Gamma \vdash \Theta}\text{ L-MO} \qquad \frac{\Gamma \vdash \Theta}{\Gamma \vdash \Theta, \Lambda}\text{ R-MO}$$

Transitivity/Cut and **Reflexivity**

$$\frac{\Gamma \vdash \Theta, A \qquad A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta}\text{ Shared-Cut} \qquad \frac{}{\Gamma, A \vdash A, \Theta}\text{ Reflexivity}$$

Contraction

$$\frac{A, A, \Gamma \vdash \Theta}{A, \Gamma \vdash \Theta}\text{ L-Contr.} \qquad \frac{\Gamma \vdash \Theta, A, A}{\Gamma \vdash \Theta, A}\text{ R-Contr.}$$

expressivism is the view that we should understand the meaning of logical connectives in terms of their expressive role. In light of the precisification, I show that the connectives introduced in my formal semantics are expressive in exactly this sense: they allow us to express *exactly* what follows from what. Following this I also show a second sense in which the machinery can be said to be expressive: we are able to mark in the object language where structural rules hold. This result not only advances the thesis of logical expressivism, but it also helps to show the tractability of the formal project. I allow that implication be non-monotonic, for example, but we are able to express where monotonicity *does* hold by the use of a sentential operator ($\boxed{M}$). One particularly interesting feature that I show can be marked in this way is captured by the natural language locutions "literally" or "strictly speaking". I show that we may introduce an operator into the object language ($\boxed{L}$) which express when implications are *fully* structural (i.e. monotonic, transitive, contractive, and reflexive). My claim is that the notion of content recaptured by this connective corresponds to more traditional understandings of content. $\boxed{L}p$ is the "minimal proposition" expressed by p.[2]

Finally, I examine attempts to account for the non-monotonicity of implication in terms of defeasible reasoning (and thus in terms of defeaters). I argue that we simply cannot account for the non-monotonicity of reasoning in these terms because we are unable to produce an account of defeaters or the phenomenon of defeat. Thus, I advocate that we should think of this phenomenon in terms of how a set of considerations hangs together; that is, in terms of the non-monotonicity of reasoning. This is meant to form a partial argument in favor of going substructural.

Having shown that we can construct a tractable formal semantics of meaning in terms of contribution to good implication and having shown why we ought to understand this phenomenon in terms of its substructurality, I next articulate what is distinctive of this approach (Ch. 3). By understanding meaning in terms of contribution to good implication, I argue that we must *push the substructurality all the way down into the content*. I argue that many other substructural logics fail to do this and thus do not take the substructurality

[2]This result will be particularly interesting when considering the third part (Ch. 4) of the dissertation, where I discuss literal meaning and minimal propositions.

of implication seriously. They are committed to what I call the *Assumption of Structurality*, which is the idea that in order to account for violations of structural rules, we must postulate some further layer of content which stands in fully structural relations of implication. It is through this postulation that a sentence p may be involved in a consequence relation which is non-monotonic (to continue the example):

$$\Gamma \vdash p \quad \text{but} \quad \Gamma, \Delta \nvdash p,$$

because there is some further layer of content that p expresses. In the context of Γ, p expresses p_1 and in the context of $\Gamma \cup \Delta$, it expresses p_2. Then:

$$\Gamma \vdash p_1$$
$$\Gamma, \Delta \nvdash p_2,$$

is not an example of a violation of monotonicity.[3] If we understand the content of a sentence in terms of its contribution to good implication such that that contribution is pushed all the way down into the content, such a postulation is not needed to understand how a sentence can be involved in implications which are substructural. Along the way, I prove some interesting results concerning substructural approaches to paradoxes. In particular, I provide some insight into the relationship between non-contractive, non-transitive, and non-reflexive solutions to paradox.

Finally, in the third part (Ch. 4), having shown that we can construct a tractable formal semantics and having articulated what is distinctive of pushing the substructurality of implication all the way down into the content, I show that this idea can be used to open up logical space in diverse regions of philosophy. I look at the semantic minimalism and radical contextualism debate in the philosophy of language, and the moral particularism and moral generalism debate in meta-ethics.

The former debate concerns whether there is such a thing as the "literal meaning" of a sentence and whether that thing is *what is said* by a sentence when it is asserted. Against this contextualists argue that what is said by a sentence may vary (perhaps radically so)

[3] Provided that $\Gamma, \Delta \vdash p_1$ and $\Gamma \nvdash p_2$, but this is precisely what the postulation is intended to make room for.

from occasion to occasion. I argue that understanding the content of a sentence in terms of its contribution to good implication provides us a way of both acknowledging the complexity of behavior that a sentence may exhibit, while also allowing us to understand the sentence as expressing the same content from occasion to occasion. To do this I expose a common presupposition underlying the debate and show how denying it opens up new logical space in the debate.

In the latter debate, moral particularists and moral generalists disagree about whether normative verdicts concerning actions are governed by moral principles. A moral principle in this debate is a rule which connects features of an action—typically statable in non-normative vocabulry—with reasons for/against that action, and eventually (perhaps through further principles governing the combination of such reasons) issues normative verdicts for the action. Moral particularists argue that we cannot articulate such principles owing to the complexity of the ways in which various reasons and considerations may interact from occasion to occasion. I show that a common presupposition underlying the debate may be denied in order to open up new logical space. Part of what we must deny to get there is the idea that a consideration is only a reason if a principle is able to explain what it is a reason for or against. Instead, if we understand reasons as that with which we reason (considerations active in our reasoning), then we can embrace a substructural notion of principle and thus reason that allows us to understand how the complexity of normative verdicts may be undergirded by stable contents.

In closing, I show that there is a deep commonality between these two debates. The crude formulation of this is that moral particularism is just radical contextualism concerning normative judgments; or, that radical contextualism is just particularism concerning truth conditions. Part of what I offer is more nuance concerning this crude approximation and thus more insight.

Understanding meaning in terms of its contribution to good implication, where such implication may be radically substructural can be used to construct a tractable formal semantics. This tractable semantics understands the substructurality of implication to be itself part of the content of a sentence. Understanding the content of a sentence in this way, I hope to have shown can do some real work outside of semantics and the philosophy of logic.

I hope to show how it can do this in two disparate debates in philosophy. I believe that there is room for it to make contributions in a number of other areas.

2.0 Meaning as Contribution to Good Implication

This chapter puts forward a candidate for the meaning of a sentence. The goal is to see what this candidate has going for it and how far it can take us. In particular, I put forward an *inferentialist* theory of meaning for sentences. This is a theory of meaning according to which the content of a sentence is be identified with that sentence's role in reasoning; i.e., the role it plays as the premise and conclusion of arguments. The account I want to explore takes this identification quite seriously: we simply understand a sentence in terms of this role.

Further, I believe that the implications in which a sentence is involved are substructural: meaning they have the potential to violate rules of inference that are often taken to be constitutive (of inference or consequence). I have in mind such structural features as monotonicity: if B follows from A, then B follows from A together with an arbitrary additional sentence C. In symbols:

$$A \vdash B \Rightarrow A, C \vdash B.$$

I also have in mind transitivity: if C follows from B and B follows from A, then C must also follow from A.

$$(A \vdash B \text{ and } B \vdash C) \Rightarrow A \vdash C.$$

Tarski, for example, understands consequence in terms of at least these two notions.[1] When I say that I want to investigate a notion of implication that is substructural, what I mean is that sentences ought to be able to stand in relations of consequence that potentially violate these rules. And they ought to be able to violate these rules in potentially radical ways.

This chapter advances such a semantic project. I start by providing a little motivation concerning substructural logic and explaining what it means to develop a theory of meaning for sentences. Following this I introduce inferentialism and my formal semantics according to which the meaning of a sentence is to be understand in terms of its contribution to good implication. Because the theory of meaning I develop involves radical substructurality—

[1] The other two structural rules that are baked into Tarski's definition are contraction and reflexivity, but more on these later.

radical here means that violations of structural rules are not accounted for by some "deeper" feature of the content—it might seem that the theory of content I develop isn't particularly tractable. In order to show that the semantics is tractable, I:

- Show that a fairly simple sequent calculus is sound and complete with respect to it.
- Develop a representation theorem for reproducing theories (in particular logical theories) with said sequent calculus.
- Along the way, I show how the formal machinery developed can help make sense of a topic of importance for inferentialists: logical expressivism.

Finally, in the close of the chapter, I provide an argument for going substructural. In particular, I try to show that attempts to account for violations of monotonicity in terms of "defeasible reasoning" don't get things right. I do this by arguing that accounts of defeat (in e.g. epistemology) fail to successfully isolate the target notion. Instead, I argue, we should understand violations of monotonicity in terms of how various considerations hang together. But the way that considerations hang together is wildly unpredictable. If this is right, then an account which allows us to understand non-monotonic implication without appeal to some "deeper" features would be a good thing indeed.

2.0.1 Substructural Logic

This dissertation is concerned with substructural logics. I should therefore take some time to explain what the "structure" in substructural refers to and thus what the "sub" in substructural refers to. Structural logic is a branch of proof theory concerned with so-called "structural" rules of proof. Its main concern is with sequent calculi, but the questions motivating this branch of logic can already be seen in studies of natural deduction. For example, if we have a proof,

$$\frac{\dfrac{A}{\vdots}}{C}$$

which begins with the undischarged assumption A and ends with C, we typically say that on the basis of A, we can prove C (or A proves C, $A \vdash C$ for short). But this informal notion is rather vague. For example, colloquially we might think that "on the basis of A and B, we can prove C" (since we can prove it from A), hence $A, B \vdash C$.[2] We might therefore want to distinguish a sense in which $A \vdash C$ but $A, B \nvdash C$,[3] since B is irrelevant[4] to the proof.[5]

Likewise, we might have a proof:

$$\dfrac{\dfrac{A}{\vdots} \qquad \dfrac{A}{\vdots}}{C}$$

It seems that $A, A \vdash C$, and there's clearly a colloquial sense in which, "on the basis of A, we can prove C", but this colloquial sense fails to mark that we've relied on A in two places. Hence, we might want to be careful distinguish that $A, A \vdash C$ but $A \nvdash C$. This is important if A can be used in ambiguous or divergent ways and we might want to keep track of that.

Particularly clear examples of this can be found among the semantic paradoxes. Paradoxes of the general sort I have in mind tend to treat the same object in divergent ways and this treatment can be easily missed if we allow ourselves to ignore cases where this happens. For example, consider the liar paradox. For the purposes of the example all that is important is that we allow ourselves to substitute α with $\tau(\alpha)$ for any α (τ is our truth predicate) as well as the following sort of substitution (λ is the liar sentence):[6]

$$\lambda = \tau(\neg\lambda).$$

Now, suppose we reason:

$$\dfrac{\lambda \qquad \dfrac{\dfrac{\lambda}{\tau(\neg\lambda)}}{\neg\lambda}}{\bot}$$

[2]It is conventional to use commas to separate premises in sequent calculi. This is especially important in structural proof theory, since we want to be careful to distinguish the kinds of structure that a collection of premises has from the kind of structure that e.g. a *set* of premises has (since sets come with their own structural assumptions built in).

[3]The '$\nvdash$' is shorthand in the meta-theory for: "it is not the case that $\ldots \vdash$ ___".

[4]I don't intend to advance a project in relevance logic. I am using what I take to be intuitive examples to help make intelligible what it means to relax these structural rules.

[5]The same questions can be raised for individual connectives. I.e. can we introduce a conditional by "discharging" a non-existent assumption?

[6]I also am going to assume standard natural deduction rules for negation and '$\bot$':

$$\dfrac{\dfrac{(n)\ \dfrac{[A]}{\vdots}}{\bot}}{\neg A}\ \neg\mathrm{I}\ (n) \qquad\qquad \dfrac{A \qquad \neg A}{\bot}\ \bot\mathrm{I}$$

A question we might ask is whether $\lambda \vdash \bot$. In a sense this seems right, but we might want to be careful to note that λ is playing two roles in this proof. In the first case it stands for some sentence, in the other case, we have substituted it multiple times (we are relying on its definitional equivalence and not on its status as just some sentence like any other). This is common to paradoxical reasoning. We often end up treating the same thing as just a mere sentence (like any other) or as something thicker: what it is defined as. If we want to mark this, we might want to be careful to write $\lambda, \lambda \vdash \bot$ since λ's use is potentially equivocal. If we allow ourselves (in the natural deduction setting) to simultaneously discharge both instances of λ, then we can derive $\neg\lambda$ (a surprising result). If we allow a few other minor substitution principles (that we can substitute λ for its definiens even when embedded in a larger sentence and double-negation elimination), then we can derive the following result (which relies on two instances of dual discharge):

$$\dfrac{\dfrac{(1)\,\dfrac{[\neg\lambda] \qquad \dfrac{\dfrac{(1)\,\dfrac{[\neg\lambda]}{\neg(\tau(\neg\lambda))}}{\neg(\neg\lambda)}}{\lambda}}{\bot}}{\neg\neg\lambda}\;\neg\mathrm{I}\ (1) \qquad \dfrac{(2)\,\dfrac{[\lambda] \qquad \dfrac{(2)\,\dfrac{[\lambda]}{\tau(\neg\lambda)}}{\neg\lambda}}{\bot}}{\neg\lambda}\;\neg\mathrm{I}\ (2)}{\bot}$$

Here is the paradigmatic problematic proof. From the definition of the liar ***alone*** (i.e. with no further undischarged premises) *follows* absurdity (i.e. $\bot$). Traditional accounts might try to deny some principle we've appealed to (i.e. that we may have self-referential definitions, or a truth predicate, etc.), but clearly the liar means something. The paradox itself is proof that we know what the sentence means in virtue of the fact that we reason with it. The problematic move, then, might just be that we have invoked λ (and $\neg\lambda$) equivocally and then glossed over that equivocation by discharging both instances simultaneously—note that if we don't allow dual discharge, the proof is simply $\neg\lambda, \lambda \vdash \bot$ (that contradictions imply falsum). We might want to therefore be careful to keep track of how many times we invoke a premise in order to guard against invoking it equivocally.[7]

Finally, suppose we have a proof from A to C, and then from C together with B to D:

[7] Another way of putting the point is that there's a choice point here. Either λ is the sort of thing that the meaning of a sentence could be (in which case we should deny contraction), or it is not (in which case we can keep contraction, but we have to explain why it is that we think it means something; how is it different from gibberish?).

$$\frac{\dfrac{A}{\vdots}}{C} \qquad\qquad\qquad \frac{\dfrac{B \quad C}{\vdots}}{D}$$

Bracketing the previous issue, would it be okay to say that we have a proof from A and B to D ($A, B \vdash D$)? In particular, couldn't we just 'cut and paste' the proofs together like this:

$$\frac{\dfrac{B \quad \dfrac{\dfrac{A}{\vdots}}{C}}{\vdots}}{D}$$

Again, we might think this is fine. Especially since we rarely encounter cases where we couldn't just prove D straightaway from A, B. But mathematicians might want to note that we took a detour through C, especially if our aim is the study of proofs, we might want to distinguish a "direct" proof, from a proof that must first establish some powerful result (C) and then use that result to derive the intended result. Perhaps that powerful result is itself controversial, or, again, we might simply want to mark such a difference.[8] Statistical and probabilistic reasoning often has this form (it may be that A makes B likely, B makes C likely, but A does not make C likely).

In general, however, what a denial of cut amounts to is the idea that a sentence might not *do* the same thing (or play the same role) when it figures as the conclusion of an argument and as the premise of an argument (insistence on "cut" amounts to denying this possibility). For example, suppose that $A =$ "b is a bird" and $C =$ "b can fly" and we allow $A \vdash C$ (that C follows from A). Now suppose that $B =$ "b lives in Antarctica" and $D =$ "b is a penguin". Clearly we should also allow that $B, C \vdash \neg D$ since if b can fly and is from Antarctica it can't

[8]We can similarly sneak in other structural constraints from this one via connective rules. For example, maybe we don't like saying that $A, B \vdash C$ if B doesn't figure in the proof, but we are fine with our conditional behaving this way. Then just let C be $B \supset A$ and now B seems to play no actual role in the following proof (I suppose we have a way to reason from A to D):

$$\frac{\dfrac{\dfrac{B \quad \dfrac{A}{B \supset A}}{A}}{\vdots}}{D}$$

The problem is that while $A \vdash B \supset A$ and $B, B \supset A \vdash C$ are both fine, $A, B \vdash D$ conceals an important middle step and there is no way to get this proof without that middle step unless one allows irrelevant premises.

Monotonicity

$$\frac{\Gamma \vdash \Theta}{\Delta, \Gamma \vdash \Theta}\ \text{L-MO} \qquad\qquad \frac{\Gamma \vdash \Theta}{\Gamma \vdash \Theta, \Lambda}\ \text{R-MO}$$

Transitivity/Cut

$$\frac{\Gamma \vdash \Theta, A \qquad A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta}\ \text{Shared-Cut}$$

Contraction

$$\frac{A, A, \Gamma \vdash \Theta}{A, \Gamma \vdash \Theta}\ \text{L-Contr.} \qquad\qquad \frac{\Gamma \vdash \Theta, A, A}{\Gamma \vdash \Theta, A}\ \text{R-Contr.}$$

Figure 2: Structural Rules

be a penguin. But cutting and pasting these together we get the result that: $A, B \vdash \neg D$, which we might not want. That b is not a penguin hardly seems to follow from its being a bird from Antarctica (its being such a bird might in fact be the most salient assumption).

The above three examples concern what are called monotonicity, contraction, and transitivity (here: shared-cut) in structural logic. If we use '$\vdash$' to encode proof, then they become the conditions on '$\vdash$' depicted in Figure 2 (the horizontal line indicates that the sequent below the line holds in case the sequent above the line holds) Substructural logics seek to deny some or all of these or to investigate under what conditions they should be suppressed or what results when they do. My suggestion is to investigate what effects such structural rules have on our understanding of content, generally, and whether we can furnish theories of substructural content.

2.1 Semantics

To start it's worth saying a few remarks about "the meaning of a sentence". Lewis (1970) famously distinguished between two senses in which we can offer a theory of meaning:

> "I distinguish two topics: first, the description of possible languages or grammars as abstract semantic systems whereby symbols are associated with aspects of the world; and, second, the description of the psychological and sociological facts whereby a particular one of these abstract semantic systems is the one used by a person or population. Only confusion comes of mixing these two topics." (Lewis, 1970, p. 19)

The first topic is typically called a "theory of meaning", the second a "meaning theory".[9] Meaning theories describe a particular interpretation of a particular language according to which items in that language are assigned meanings in a way that preserves some structure of the language. A theory of meaning says what form a meaning-theory must take (or sets some general constraints on meaning-theories). That is: a theory of meaning tells us what the elements of a language are assigned to, and what the relevant structure is. In this dissertation I provide a theory of meaning.

I provide a theory of the meaning of *sentences* for several reasons.[10] First, I agree with Frege's context principle that the constituents of a sentence acquire their content through their appearance in sentences (emphasis mine):

> "Accordingly, any word for which we can find no corresponding mental picture appears to have no content. But we ought always to keep before our eyes a complete proposition. *Only in a proposition have the words really a meaning.* It may be that mental pictures float before us all the while, but these need not correspond to the logical elements in the judgment. It is enough if the proposition taken as a whole has a sense; it is this that confers on its parts also their content." (Frege, 1980, §60)

It is only in the context of a sentence that words acquire their content. Thus, if one can put forward an account of the content of a sentence, then the meanings of sub-sentential expressions ought to be downstream from this.[11]

[9] As far as I can tell these terms originated with Dummett (1975, 1976). Their use is fairly ubiquitous now.

[10] Not least of which is that getting the sentential right is difficult enough. I hope to develop the work in this dissertation for quantification and sub-sentential expressions.

[11] Though this oversimplifies the matter.

Next, we ought to take the content of a sentence to be connected in some way with the (or some privileged) characteristic *use* of that sentence. For example, one (or perhaps *the*) basic function of language is communication. The most basic communicative act is taken to be assertion. So whatever content we express by asserting a sentence is what we should understand the meaning of the sentence to be.

Here there emerges a large divide among theories of meaning. Some understood assertion in terms of something like a norm for truth.[12] In that case, we should understand the content of a sentence in terms of its potential for truth/falsehood. According to this view, then, the content of a sentence is to be identified with its truth conditions (i.e. those circumstances under which it is true).[13]

In contrast to this, however, we might instead understand assertion not in terms of norms for its proper performance, but in terms of what we accomplish through assertion. Lewis (1979); Brandom (1983) suggest understanding assertion in terms of making a move in a language game; in particular the undertaking of a commitment. If we have a rational conception of language and communication, then the commitment is a commitment to justifying one's claim if challenged. According to such a conception of language, an essential aspect of a sentence's meaning is how it figures in reasoning: the sorts of claims it is capable of supporting under challenge and what sorts of further claims could help support its assertion (on challenge). What sorts of claims a sentence is capable of supporting, we should understand in terms of what consequences the sentence has, and what sorts of claims are capable of supporting a sentence, we should understand as those sentences which have the original sentence as a consequence. Taking these two aspects of a sentence to be constitutive of the sentence's meaning gives us a particular theory of meaning: an inferentialist theory of meaning.

[12] I am oversimplifying the matter. In particular, I am limiting my attention to normative conceptions of assertion. Many understand assertion, for example, in terms of the expression of a belief (or some other mental state). Then the meaning of a sentence corresponds to the structure of that mental state. Or perhaps the sentence simply refers to such a mental state (if we individuate mental states functionally, such a proposal is not that far fetched). At any rate, this takes me rather far afield from where I hope to go.

[13] I should note that when it comes to normative accounts of assertion, understanding assertion in terms of *knowledge* is a much more popular proposal. See Williamson (1996), who popularized this proposal. That assertion should be understood in terms of a norm for *truth* is put forward by MacFarlane (2011).

2.1.1 Inferentialism

Inferentialism is the view that the role that something (paradigmatically, a sentence) plays in reasoning, e.g. the role it serves as a premise and conclusion of arguments, is an essential element of that thing's meaning. In the central case, the meaning of a sentence is given by its role as a premise and a conclusion in argument, or as I shall say: the meaning of p just is the contribution that p makes to the goodness of implication. Thus, we might understand the meaning of p as specified in:

$$\begin{array}{ccc} p, \Gamma_1 \vdash \Theta_1 & \qquad\qquad & \Gamma_1 \vdash \Theta_1, p \\ p, \Gamma_2 \vdash \Theta_2 & & \Gamma_2 \vdash \Theta_2, p \\ \vdots & & \vdots \\ p, \Gamma_n \vdash \Theta_n & & \Gamma_n \vdash \Theta_n, p \\ \vdots & & \vdots \end{array}$$

On an inferentialist understanding of meaning, we treat the meaning of p as the contribution it makes to good inference above. This immediately introduces two constraints.

(Constraint One) The first constraint is that a sentence is only meaningful if it has a role as a premise and as a conclusion, (i.e. we must specify *two* lists). It's important that we specify two roles for at least two reasons. First, two sentences may play more-or-less the same role as a premise (or as a conclusion) but play different roles as conclusions. Second, the idea that a sentence might appear as a conclusion but never as a premise (or vice-versa) is unintelligible if we understand what sentences express to be rationally related to other sentences. We can appreciate both of these points clearly by considering two concepts that are extensionally equivalent (but differ intensionally: or two sentences with equivalent truth conditions, which express different senses, if one prefers to keep things sentential).

I introduce the following shorthand to talk about the contribution that, e.g. a sentence, makes to good implication. Since we'll need a way of noting the role that a sentence plays as a premise and as a conclusion, we'll have to note which one we're after:

$$\langle\{p\},\emptyset\rangle^{\curlyvee} =_{df.} \{\langle\Gamma,\Theta\rangle | p,\Gamma \vdash \Theta\}, \qquad (p \text{ as premise})$$

$$\langle\emptyset,\{p\}\rangle^{\curlyvee} =_{df.} \{\langle\Gamma,\Theta\rangle | \Gamma \vdash \Theta, p\}. \qquad (p \text{ as conclusion})$$

which specify the contribution that p makes as a premise and conclusion, respectively, to the goodness of implication. Putting it all together then, I use double-brackets, '$[\![\cdot]\!]$" to denote the contribution of p in total:

$$[\![p]\!] =_{df.} = \langle\langle\{p\},\emptyset\rangle^{\curlyvee}, \langle\emptyset,\{p\}\rangle^{\curlyvee}\rangle.$$

We might think of this as shorthand for the contribution to good implication that p made in the lists above. This constraint is equivalent in the valuational setting to the idea that a sentence needs specifiable truth-conditions. According to that account you can say many things about a sentence or how it might be spoken truly, but if you haven't specified truth conditions, you haven't specified its meaning.

The second constraint (below) also has a valuational correlate. This is the constraint that if two sentences have the same truth conditions then they are equivalent: that is, we must be extensional concerning truth conditions. To be precise: sentences are equivalent if they are true on exactly the same models. This constraint will end up imposing a kind of extensionality concerning meaning. This is especially important because we don't want to allow one sentence to have multiple *contents*; nor to allow *distinct* sentences the same content. The second constraint therefore fixes sentence individuation at the level of contribution to good implication.

(Constraint Two) A contribution to good implication fixes an equivalence class among inferential roles. Above I said that we had to distinguish two "roles" that a sentence plays in implication (two separate contributions it makes), but what counts as a role? I gave two lists, but is that what it means to be a role? Are there any constraints on what counts as contributing to the goodness of implication as either a premise or conclusion? Because I'm interested in relaxing structural rules of implication (in particular, contraction), I intend to be fairly liberal as regards what could count as a role in good implication. Even if I allow myself maximal liberality, however, one constraint still arises: contributions are equivalence classes of implicational roles. The intent of this is to fix equivalence of meaning at a certain level; namely the ground level: in terms of contribution to good implication. p (as a premise) and q (as a premise) are equivalent iff they make the same contribution to good implication.

As it turns out, the notions I've introduced so far allow us to define an intricate inferential role semantics. We need only enrich it with one additional operation, which I call adjunction. I shall now introduce the formal details. The semantics gives us a natural and intuitive way to think about the meaning of a sentence. Further, I produce fairly straightforward semantic clauses for connectives and these turn out to correspond to standard sequent calculus presentations of those connectives.[14]

2.1.2 The Formal Details

Definition 2.1.1 (Inferential Space $\mathbf{P}$, and Good Implications $\mathbb{I}$)**.** Let $\mathcal{L}$ be our language (of potential logical complexity) For my purposes here $\mathcal{L}$ is a propositional language, but there are natural extensions to first-order languages. An **inferential space** is the set of all ordered pairs of *multi-sets* of $\mathcal{L}$: $\mathbf{P} = \mathcal{P}(\mathcal{L})^2$. We call each "point" (of the form $\langle X, Y \rangle$, where $X, Y \subseteq \mathcal{L}$) an **implication**. Each inferential space $\mathbf{P}$ comes with a privileged subset of implications: the **good implications**: $\mathbb{I} \subseteq \mathbf{P}$.

Definition 2.1.2 (Adjunction)**.** There is a single associative and commutative operation on $\mathbf{P}$ called **adjunction**, '$\sqcup$'. If $A = \langle \Gamma, \Theta \rangle$ and $B = \langle \Delta, \Lambda \rangle$, then

$$A \sqcup B =_{df.} \langle \Gamma \cup \Delta, \Theta \cup \Lambda \rangle.$$

We also generalize '$\sqcup$' as an operation over subsets of $\mathbf{P}$.[15] If $X, Y \subseteq \mathbf{P}$, then:

$$X \sqcup Y = \{x \sqcup y | x \in X, y \in Y\}.$$

Next, I'll give a formal definition (aided by my recent definition of adjunction) of '$\curlyvee$' (pronounced "vee"). '$\curlyvee$' specifies the role that an ordered pair plays in good implication, i.e. which pairs in can be adjuncted such that the result is a good implication.

[14]The semantic picture below is heavily inspired by—in some sense isomorphic to a mere variation on—Girard's phase space space semantics. See (Girard, 2011).

[15]Recall that '$\cup$'—as in '$\Gamma \cup \Theta$'—is multi-set-union. Hence it may be that $\Gamma \cup \Gamma \neq \Gamma$). We treat the set of points as a *set* and hence subsets of $\mathbf{P}$ as sets of points—even if those points are structured by multi-sets.

Definition 2.1.3 (vee)**.** Suppose $X \subseteq \mathbf{P}$. Then:[16]

$$X^{\curlyvee} =_{df.} \{\langle \Delta, \Lambda \rangle \,|\, \forall \langle \Gamma, \Theta \rangle \in X\, (\langle \Gamma, \Theta \rangle \sqcup \langle \Delta, \Lambda \rangle \in \mathbb{I})\}.$$

With '$\curlyvee$' in hand, we can define when a set of implications is closed. Recall that earlier I said a second constraint on meaning is that contributions form equivalence classes of roles. This is because we wanted to fix extensionality of meaning at the level of contribution to inferential role. And it is easy to see that this defines a closure operation—since $(\cdot)^{\curlyvee}$ can be used to define a symmetric relation on subsets of $\mathbf{P}$, $(\cdot)^{\curlyvee\curlyvee}$ defines an equivalence relation.

Definition 2.1.4 (Closure)**.** A set of implications $X \subseteq \mathbf{P}$ is said to be **closed** iff $X^{\curlyvee\curlyvee} = X$.

Proposition 2.1.5. $(\cdot)^{\curlyvee\curlyvee}$ is a closure operation, i.e. $(\cdot)^{\curlyvee\curlyvee}$ is **extensive** ($X \subseteq X^{\curlyvee\curlyvee}$), **idempotent** ($X^{\curlyvee\curlyvee\curlyvee\curlyvee} = X^{\curlyvee\curlyvee}$) and **monotone** (if $X \subseteq Y$, then $X^{\curlyvee\curlyvee} \subseteq Y^{\curlyvee\curlyvee}$).

This closure operation allows us to introduce what I shall call "proper inferential roles" (PIRs). These are pairs of closed sets of implications. Proper inferential roles will serve as the interpretants of sentences. What is distinctive of *proper* inferential roles is that they satisfy both of the constraints that I rehearsed in the previous section.

Definition 2.1.6 (Proper Inferential Role)**.** A **proper inferential role (PIR)** is an ordered pair $\langle X, Y \rangle$ such that X and Y are each *closed*—in the sense defined above—subsets of $\mathbf{P}$ (i.e. $X^{\curlyvee\curlyvee} = X$ and $Y^{\curlyvee\curlyvee} = Y$).

When we think of $\langle X, Y \rangle$ as the semantic interpretant of a sentence A, we think of $\langle X, Y \rangle$ as consisting of a part, X, that specifies the role that A plays as a premise and a part, Y, that specifies the role A plays as a conclusion. The set of implications X are those implications into which A can be added as a premise to get a good implication. And Y is the set of implications into which A can be added as a conclusion to get a good implication. We say that X is the premissory role and Y the conclusory role of A.

[16] In the previous sub-section I defined '$\curlyvee$' over single ordered pairs. That definition, with the aid of adjunction would appear as follows (i.e. if X were a singleton—so suppose $X = \{\langle \Gamma, \Delta \rangle\}$):

$$X^{\curlyvee} = \{\langle \Gamma, \Delta \rangle\}^{\curlyvee} =_{df.} \{\langle \Delta, \Lambda \rangle | \langle \Gamma, \Delta \rangle \sqcup \langle \Delta, \Lambda \rangle \in \mathbb{I}\}.$$

Definition 2.1.7 (Convention)**.** As a convention if $[\![A]\!] = \langle X, Y\rangle$ is an inferential role, then we write $[\![A]\!]_P$ to refer to X and $[\![A]\!]_C$ to refer to Y, i.e. A's premissory and conclussory roles, respectively.

Philosophically, we should understand PIRs as specifying absolutely minimal, inferentialist constraints that semantic content must satisfy in accordance with Constraints One and Two above. In particular, in order to have specified the content of a sentence, (One:) one must specify both a premissory and a conclusory role for that content; and (Two:) those premissory and conclusory inferential roles must be closed, i.e., they must be sufficiently robust so that two roles cannot be merely notionally differentiated. For example suppose $X^{\curlyvee} = Y^{\curlyvee}$ (intuitively, X and Y play the same role in good implication), but $X \neq Y$. Then neither X nor Y is closed, since it is easy to show that $Y \subseteq X^{\curlyvee\curlyvee}$ and $X \subseteq Y^{\curlyvee\curlyvee}$.

Definition 2.1.8 (Models)**.** A **model** is a quadruple $\langle \mathcal{L}, \mathbf{P}, \mathbb{I}, [\![\cdot]\!]\rangle$ consisting of a language $\mathcal{L}$ and inferential space over that language $\mathbf{P}$, a privileged set of good implications $\mathbb{I}$, and an interpretation function $[\![\cdot]\!]$ (to be defined next) which interprets sentences in the language as inferential roles in the model.

Definition 2.1.9 (Interpretation Function)**.** An interpretation function $[\![\cdot]\!]$ maps sentences in $\mathcal{L}$ to proper inferential roles in models. If $A \in \mathcal{L}$ is atomic, then A is interpreted as follows:[17]

$$[\![A]\!] =_{df.} \langle \langle \{A\}, \emptyset\rangle^{\curlyvee}, \langle \emptyset, \{A\}\rangle^{\curlyvee}\rangle.$$

The semantic definitions of connectives follows:[18]

$$\begin{aligned}
[\![A\&B]\!] &=_{df.} \langle (([\![A]\!]_P)^{\curlyvee} \sqcup ([\![B]\!]_P)^{\curlyvee})^{\curlyvee}, [\![A]\!]_C \cap [\![B]\!]_C\rangle,\\
[\![A \vee B]\!] &=_{df.} \langle [\![A]\!]_P \cap [\![B]\!]_P, (([\![A]\!]_C)^{\curlyvee} \sqcup ([\![B]\!]_C)^{\curlyvee})^{\curlyvee}\rangle,\\
[\![A \to B]\!] &=_{df.} \langle [\![A]\!]_C \cap [\![B]\!]_P, (([\![A]\!]_P)^{\curlyvee} \sqcup ([\![B]\!]_C)^{\curlyvee})^{\curlyvee}\rangle.,\\
[\![\neg A]\!] &=_{df.} \langle [\![A]\!]_C, [\![A]\!]_P\rangle.
\end{aligned}$$

Definition 2.1.10 (Semantic Entailment)**.** We say that A semantically entails B relative to a model $\mathcal{M}$ if the closure of the combination of A (as premise) and B (as conclusion)

[17]Note that, for arbitrary X, $X^{\curlyvee}$ is closed. Hence, we are guaranteed that $[\![A]\!]$ will be a PIR.

[18]I only introduce so-called "negative" connectives here, i.e. connectives with rules that are reversible.

consists of only good implications:

$$A \vDash_{\mathcal{M}} B \quad \text{iff}_{df.} \quad ((\llbracket A \rrbracket_P)^{\curlyvee} \sqcup (\llbracket B \rrbracket_C)^{\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathbb{I}_{\mathcal{M}}.$$

We say that A semantically entails B if $A \vDash_{\mathcal{M}} B$ on all models $\mathcal{M}$.

NB: If A and B are multi-sets of sentences—suppose $A = \{A_1, \ldots, A_n\}$ and $B = \{B_1, \ldots, B_m\}$—then we read $A \vDash B$ as, for all models $\mathcal{M}$:

$$A_1, \ldots, A_n \vDash_{\mathcal{M}} B_1, \ldots, B_m \quad \text{iff}_{df.}$$
$$((\llbracket A_1 \rrbracket_P)^{\curlyvee} \sqcup \cdots \sqcup (\llbracket A_n \rrbracket_P)^{\curlyvee} \sqcup (\llbracket B_1 \rrbracket_C)^{\curlyvee} \sqcup \cdots \sqcup (\llbracket B_m \rrbracket_C)^{\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathbb{I}_{\mathcal{M}}.$$

I will need to introduce more machinery, before I can prove anything interesting about entailment. But what I want to note here is the natural thought that what it means to say that A entails B (that B semantically follows from A) is nothing other than to say that the adjunction of A's premissory role and B's conclussory contains nothing but good implications. In the case where A and B are multi-sets, it is instead the adjunction of all premises (qua premises) and all conclusions (qua conclusions).

2.1.2.1 Generating Consequence Relations

In this section I explain how we may generate (substructural) consequence relations in line with the inferentialist view of meaning I sketched above.

Definition 2.1.11 (Base Consequence Relation (BCR))**.** A base consequence relation is a relation between finite multi-sets of atomic sentences, e.g. $\vdash_0 \subseteq \mathcal{P}(\mathcal{L}_0)^2$ (where $\mathcal{L}_0$ is the set of all atomic sentences of the language).

We start with what I call a "material" base consequence relation, i.e. a base consequence relation with no further constraints on it.

To be clear, I require none of the following constraints on a base consequence relation:

Definition 2.1.12 (Constraints on a base)**.** A base consequence relation may satisfy certain constraints.

Reflexive: A BCR is **reflexive** iff, for all $p \in \mathcal{L}_0$: $p \vdash_0 p$.

Axiom: If $\Gamma \vdash_0 \Theta$ then $\Gamma \vdash \Theta$.

$$\frac{\Gamma \vdash \Theta, A \qquad B, \Gamma \vdash \Theta}{A \to B, \Gamma \vdash \Theta}\ \text{L}\to \qquad\qquad \frac{A, \Gamma \vdash \Theta, B}{\Gamma \vdash A \to B, \Theta}\ \text{R}\to$$

$$\frac{\Gamma, A, B \vdash \Theta}{\Gamma, A\&B \vdash \Theta}\ \text{L\&} \qquad\qquad \frac{\Gamma \vdash \Theta, A \qquad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A\&B}\ \text{R\&}$$

$$\frac{A, \Gamma \vdash \Theta \qquad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta}\ \text{L}\vee \qquad\qquad \frac{\Gamma \vdash \Theta, A, B}{\Gamma \vdash \Theta, A \vee B}\ \text{R}\vee$$

$$\frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta}\ \text{L}\neg \qquad\qquad \frac{A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta, \neg A}\ \text{R}\neg$$

Figure 3: Sequent Calculus

Containment: A BCR satisfies **containment** iff, for all $p \in \mathcal{L}_0$ all $\langle \Delta, \Lambda \rangle \in \mathcal{P}(\mathcal{L}_0)^2$ we have: $\Delta, p \vdash_0 p, \Lambda$.

Monotonicity: A BCR is **monotonic** iff for all $\langle \Gamma, \Theta \rangle$ and $\langle \Delta, \Lambda \rangle$ in $\mathcal{P}(\mathcal{L}_0)^2$ we have: if $\Gamma \vdash_0 \Theta$ then $\Delta, \Gamma \vdash_0 \Theta, \Lambda$.

Contractive: A BCR is **contractive** iff $A, A, \Gamma \vdash_0 \Theta$ only if $A, \Gamma \vdash_0 \Theta$ and $\Gamma \vdash_0 \Theta, A, A$ only if $\Gamma \vdash_0 \Theta, A$ (for arbitrary Γ, Θ, A).

Transitive: A BCR is **transitive** iff $A, \Gamma \vdash_0 \Theta$ and $\Gamma \vdash_0 \Theta, A$ only if $\Gamma \vdash_0 \Theta$ (for arbitrary Γ, Θ, A.[19]

Next, I define a consequence relation, $\vdash$ as any sequent derivable from the sequent calculus in Figure 3, where the leaves are generated from the following single axiom (given a particular BCR $\vdash_0$).

2.1.2.2 Soundness and Completeness

Since we haven't imposed any restriction on models, however, it is easy to find counter-models for our base consequence relation. Thus, we need to appropriately limit models based

[19]We can define a similar version for mixed-cut. I do not in this document.

upon what $\vdash_0$ looks like.

Definition 2.1.13 (Base Consequence Relation)**.** A **base consequence relation** is a subset of $\mathbf{P}$ that consists of only atoms. B is a base consequence relation iff $B \subseteq \mathbf{P}$ and $B \cap \mathcal{P}(\mathcal{L}_0)^2 = B$.

We say that a model $\mathcal{M} = \langle \mathbf{P}, \mathbb{I}, [\![\cdot]\!] \rangle$ is **fit for** a base consequence relation B iff

$$\forall \langle \Delta, \Lambda \rangle \in B(\Delta \vDash_{\mathcal{M}} \Lambda).$$

We say that Γ semantically entails Θ **relative to** B iff $\Gamma \vDash_{\mathcal{M}} \Theta$ for all models $\mathcal{M}$ that are fit for B. We write this as $\Gamma \vDash_B \Theta$.

We are now in a position to show that the sequent calculus introduced in the previous section is sound and complete with respect to the inferential role semantics.

Theorem 2.1.14 (Soundness)**.** *The sequent calculus is sound:*

$$\Gamma \vdash_B \Theta \Rightarrow \Gamma \vDash_B \Theta.$$

Proof. The proof proceeds via induction on proof height. The base case is guaranteed by our restriction to models fit for B. To illustrate the proof, we do two interesting cases; the other cases are analogous. If the last step in our proof tree is R& (and so our sequent is $\Gamma \vdash \Theta, A\&B$), then we have $\Gamma \vDash_B \Theta, A$ and $\Gamma \vDash_B \Theta, B$, i.e.:

$$(([\![\&\Gamma]\!]_P)^{\curlyvee} \sqcup ([\![\bigvee(\Theta \cup \{A\})]\!]_C)^{\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathbb{I}$$

and

$$(([\![\&\Gamma]\!]_P)^{\curlyvee} \sqcup ([\![\bigvee(\Theta \cup \{B\})]\!]_C)^{\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathbb{I}.$$

So:

$$(([\![\&\Gamma]\!]_P)^{\curlyvee} \sqcup ([\![\bigvee(\Theta \cup \{A\})]\!]_C)^{\curlyvee})^{\curlyvee\curlyvee} \cap (([\![\&\Gamma]\!]_P)^{\curlyvee} \sqcup ([\![\bigvee(\Theta \cup \{B\})]\!]_C)^{\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathbb{I}.$$

Since this is just the semantic definition of '&' qua conclusion, it is easy to see that we get the required semantic entailment:

$$(([\![\&\Gamma]\!]_P)^{\curlyvee} \sqcup ([\![\bigvee(\Theta \cup \{A\&B\})]\!]_C)^{\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathbb{I},$$

so:

$$\Gamma \vDash_B \Theta, A\&B.$$

Likewise, suppose the last step is R$\rightarrow$ (and so our sequent is $\Gamma \vdash \Theta, A \rightarrow B$). Then we have:

$$((\llbracket \&(\Gamma \cup \{A\} \rrbracket_P)^{\curlyvee} \sqcup (\llbracket \bigvee(\Theta \cup \{B\}) \rrbracket_C)^{\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathbb{I}.$$

Via some simple manipulations we reason:

$$((\llbracket \&\Gamma \rrbracket_P)^{\curlyvee} \sqcup (\llbracket \bigvee\Theta \rrbracket_C)^{\curlyvee} \sqcup (\llbracket A \rrbracket_P)^{\curlyvee} \sqcup (\llbracket B \rrbracket_C)^{\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathbb{I}.$$

From which it is straightforward to show that

$$((\llbracket \&\Gamma \rrbracket_P)^{\curlyvee} \sqcup (\llbracket \bigvee\Theta \rrbracket_C)^{\curlyvee} \sqcup (\llbracket A \rightarrow B \rrbracket_C)^{\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathbb{I}.$$

And thus:

$$((\llbracket \&\Gamma \rrbracket_P)^{\curlyvee} \sqcup (\llbracket \bigvee\Theta \cup \{A \rightarrow B\} \rrbracket_C)^{\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathbb{I},$$

which just means

$$\Gamma \vDash_{\mathcal{M}} \Theta, A \rightarrow B.$$

■

Theorem 2.1.15 (Completeness). *The sequent calculus is complete:*

$$\Gamma \vDash_B \Theta \Rightarrow \Gamma \vdash_B \Theta.$$

Proof. We show the contrapositive (i.e. that $\Gamma \nvdash_B \Theta \Rightarrow \Gamma \nvDash_B \Theta$) by constructing canonical models that can serve as counter-models for $\Gamma \nvdash_B \Theta$. To do this, we construct a model $\mathcal{M}$ which has the feature that $\Gamma \vDash_{\mathcal{M}} \Theta \Leftrightarrow \Gamma \vdash_B \Theta$.

It is easy to show (by induction on proof height) that we get this result if we define $\mathbb{I}_{\mathcal{M}}$ on our model as:

$$\mathbb{I}_{\mathcal{M}} =_{df.} \{\langle \Gamma, \Theta \rangle \in \mathbf{P} \mid \Gamma \vdash_B \Theta\}.$$

■

2.2 Representation Theorem & Logical Expressivism

In the previous section I produced a semantics for sentences which stand in (potentially radically) substructural relations of implication. That is, implications which may be non-monotonic, non-transitive, non-contractive, and non-reflexive. I also showed that the semantics is relatively tractable, insofar as relatively straightforward definitions of logical connectives allowed Gc3p (or Ketonen) style rules to be sound and complete for the semantics. This section compiles more evidence in favor of the claim that the semantics (and sequent calculus which is sound and complete with respect to it) is tractable and inherently interesting.

To do this, I prove a representation theorem for base consequence relations. That is, a method to get any theory (provided some modest constraints are met) via restrictions on base consequence relations. This result also gives us a way of precisifying a notion of some interest in the literature on inferentialism: namely logical expressivism.[20]

2.2.1 A Precisification of Logical Expressivism

I now seek to make the notion of "expression" more precise. Brandom understands expressivism in terms of what he calls an "LX relation", where a vocabulary B is "LX" of a vocabulary A if it is <u>el</u>aborated from and <u>ex</u>plicative of A. The first criterion (elaboration) has it that if one is able to successfully deploy vocabulary A then one already has the skills necessary to use B. That is, that B may be (algorithmetically) elaborated from A. The second criterion (explication) has it that B says something about (makes perspicuous in the object language) what one was doing by using A (minimally that B may encode the implications and incompatibilities of A). Logical vocabulary is said to be "universally LX" meaning that logical vocabulary stands in this relation to *all vocabularies.*

Let us make this relation more precise. First let $\mathcal{L}_0$ be an arbitrary vocabulary devoid of logical symbols (i.e. a set of atomic sentence letters). Let $\mathrel{\succ\!\!\sim}_0$ be a consequence relation over

[20]See Brandom (2008) for an original formulation. Peregrin (2014); Brandom (2018a,b) for some developments. See also Hlobil (2016) and Kaplan (2018). The latter of which is a predecessor to work carried out in the present section. Unpublished work by Ulf Hlobil, Ryan Simonelli, and Shuhei Shimamura has developed some of these themes.

$\mathcal{L}_0$ (i.e. $\succ_0 \subseteq \mathcal{P}(\mathcal{L}_0)^2$). Note that while I call $\succ_0$ and $\succ$ (below) *consequence* relations I do not yet impose *any* restrictions on them.[21] They should be treated, therefore, simply as two place relations between sets of sentences. As I discussed in the introduction, there are philosophically rich reasons for wanting a consequence relation that is e.g. non-monotonic or perhaps non-classical. In addition part of the motivation of expressivism is that where such features hold of consequence it is an *expression* of an underlying (material) relation of consequence.[22]

Next let $\mathbb{L}$ be our logic. Our logic consists of a finite set of logical symbols (e.g. $\{\&, \vee, \neg, \rightarrow\}$) and rules for expanding $\mathcal{L}_0$ to $\mathcal{L}$ (our language enriched with those logical symbols) and for expanding $\succ_0$ to $\succ$. Intuitively, we should think of $\mathbb{L}$ as a function from $\succ_0$ to $\succ$. That is, $\mathbb{L} : \succ_0 \mapsto \succ$. Then whether $\mathbb{L}$ is "LX" concerns the relationship between $\succ_0$ and $\succ$ (i.e. the behavior of $\mathcal{L}$ in relation to the behavior of $\mathcal{L}_0$).

That the logical vocabulary be elaborated fixes a tight relationship from $\succ_0$ to $\succ$. That is, to get from $\succ_0$ to $\succ$ should require no more than a specification of the logical vocabulary. That is, given $\succ_0$, $\succ$ should be uniquely determined: $\succ_0 \Rightarrow \succ$. In prose, the behavior of $\mathcal{L}$ should be determined by the behavior of $\mathcal{L}_0$ simply by specifying the logical symbols.

That the logical vocabulary be explicative fixes a tight relationship from $\succ$ to $\succ_0$. Since this requires that the logical vocabulary enable us to *say something* about the underlying pre-logical consequence relation, we should require that it actually do what it purports to do: $\succ \Rightarrow \succ_0$. In prose, the behavior of $\mathcal{L}$ should *genuinely* say or *express* something about the behavior of $\succ_0$. The behavior of $\mathcal{L}$ should therefore fix the behavior of $\mathcal{L}_0$. If $\mathcal{L}$ behaved differently then it would *express* something different about the behavior of $\mathcal{L}_0$. If such expression is to be genuine then the behavior of $\mathcal{L}_0$ (i.e. $\succ_0$) would need to be different.

Together these two criterion have it that $\succ_0 \Leftrightarrow \succ$. The behavior of $\mathcal{L}$ is elaborated out of, but also explicative of the behavior of $\mathcal{L}_0$.

[21] When speaking of consequence relations in a general sense, I will use "$\succ$" to formulate things. This is to make it clear that the formulation is meant more generally (i.e. it needn't be about the notion of consequence explored earlier in this chapter. I believe Lloyd Humberstone is responsible for this convention.

[22] I am being brief here on the *justification* for treating $\succ_0$ as I do. I primarily wish to stress here—in order to avoid confusion—that $\succ_0$ need not have *any constraints*.

While this criterion has some naive plausibility, it must still be made more precise. In particular, if our logical vocabulary is to be *conservative*, then $\mathrel{\succ}_0 \subseteq \mathrel{\succ}$ and so the criterion will hold trivially. We may circumvent this problem by quantifying over possible $\mathrel{\succ}_0$. This might have already been anticipated since I mentioned that logical vocabulary is to have this relationship *universally* i.e. with respect to arbitrary vocabularies (and thus arbitrary $\mathrel{\succ}_0$). This gives rise to the following definition:

Definition 2.2.1. Fix a logic $\mathbb{L}$, i.e. a function from $\mathrel{\succ}_0$ to $\mathrel{\succ}$. We say that $\mathbb{L}$ is **expressive** or that $\mathrel{\succ}$ **expresses** a base consequence relation $\mathrel{\succ}_0$ iff:

$$
\begin{gathered}
(\forall \Gamma, \Theta \subseteq \mathcal{L})(\exists \Gamma_1, \Theta_1, \ldots, \Gamma_n, \Theta_n \subseteq \mathcal{L}_0) \\
(\forall \mathrel{\succ}_0 \subseteq \mathcal{P}(\mathcal{L}_0)^2)(\forall \mathrel{\succ} \subseteq \mathcal{P}(\mathcal{L})^2(\mathbb{L} : \mathrel{\succ}_0 \mapsto \mathrel{\succ})) \\
((\Gamma \mathrel{\succ} \Theta) \Leftrightarrow (\Gamma_1 \mathrel{\succ}_0 \Theta_1 \bigwedge \Gamma_2 \mathrel{\succ}_0 \Theta_2 \bigwedge \cdots \bigwedge \Gamma_n \mathrel{\succ}_0 \Theta_n)).
\end{gathered}
$$

We also say $\Gamma \mathrel{\succ} \Theta$ **expresses** $\Gamma_i \mathrel{\succ}_0 \Theta_i$ $(1 \leq i \leq n)$ (its expressientia).

This definition says anytime $\Gamma \mathrel{\succ} \Theta$ this is in virtue of some set of implications present in the language prior to $\mathbb{L}$. So the logical vocabulary is said to be elaborated if $\Gamma \mathrel{\succ} \Theta$ occurs whenever those pre-logical implications obtain, and the logical vocabulary is said to be explicative if $\Gamma \mathrel{\succ} \Theta$ occurs only if those pre-logical implications obtain.

The above should be taken as a precise specification of a minimal constraint on logics to count as "expressive". But one of the central features of expression is the idea that logical vocabulary should be able to make perspicuous in the object language *structural features* of inference. By structural features I have in mind such things as monotonicity, transitivity, contraction, reflexivity, classicality, etc., where each is understood to be capable of holding both globally (e.g. that $\mathrel{\succ}$ is monotonic) as well as locally (e.g. that $\Gamma \mathrel{\succ} \Theta$ is monotonic, though $\Delta \mathrel{\succ} \Lambda$ may not be). Expressivism says that it is distinctive of logical vocabulary to be able to express such features. This requires that (i) $\mathbb{L}$ be capable of *preserving* structural features and that (ii) $\mathbb{L}$ be capable of *expressing* those very structural features it preserves.

Definition 2.2.2. Let $\mathfrak{Sf}$ be a structural feature. Officially, a structural feature is a property of elements of consequence relations. So we can think of $\mathfrak{Sf}$ as a family of partitions, one for each member of the family of consequence relations (i.e. a partition for each subset of

$\mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L}))$.[23] Let $\mathfrak{Sf}(\Gamma \succ\!\!- \Theta)$ be shorthand for $\Gamma \succ\!\!- \Theta$ obeys (is an instance of) $\mathfrak{Sf}$. Next, let $\Gamma \succ\!\!- \Theta$ be arbitrary with $\Gamma_i \succ\!\!-_0 \Theta_i$ $(1 \leq i \leq n)$ its expressientia (in accordance with Definition 2.2.1). We say that a logic $\mathbb{L}$ **preserves** a structural feature $\mathfrak{Sf}$ iff:

$$\mathfrak{Sf}(\Gamma \succ\!\!- \Theta) \Leftrightarrow \left(\mathfrak{Sf}(\Gamma_1 \succ\!\!-_0 \Theta_1) \bigwedge \cdots \bigwedge \mathfrak{Sf}(\Gamma_n \succ\!\!-_0 \Theta_n)\right).$$

A structural feature is *preserved* when an implication obeys that structural feature *iff* all of the implications it expresses also obey that structural feature. Thus, whether an implication obeys a structural feature should be seen as expressing something about the pre-logical implications that that implication expresses: it inherits those features from them and has those features in virtue of those implications alone. Next, I must explain what it means for a particular piece of logical vocabulary to express such structural features.

Definition 2.2.3. Let $\mathfrak{Sf}$ be a structural feature. Suppose some logical operation '*' may be used to mark a sequent in some way (with the constraint that $\Gamma^* \succ\!\!- \Theta^*$ only if $\Gamma \succ\!\!- \Theta$). Then we say that '*' (or $\mathbb{L}$) expresses $\mathfrak{Sf}$ iff there exists a '*' in $\mathbb{L}$ such that:

$$\Gamma^* \succ\!\!- \Theta^* \Leftrightarrow \mathfrak{Sf}(\Gamma \succ\!\!- \Theta).$$

Sf-Expression combines three ideas. (i) That a logic be capable of expressing an underlying base consequence relation, (ii) that it be capable of preserving structural features of that base consequence relation, and finally (iii) that it be able to mark in the object language those very same features that it preserves.

[23] Unofficially, a structural feature usually has something like an intension associated with it. It may be that the monotonic and transitive consequences of a consequence relation are the same. Nevertheless, the characterization of each is different, because the extensions may differ on a different consequence relation.

2.2.2 Representation Theorem

The logic I defined in the previous section is precisely such a logic. I repeat the details here for ease and prove some novel results surrounding the sequent calculus. Following this I show that the logic is expressive in the sense defined above by proving two representation theorems. Let us fix a base consequence relation (BCR) $\vdash_0$. Our logic include the symbols $\{\&, \vee, \neg, \to\}$ and expands $\mathcal{L}_0$ to $\mathcal{L}$ in the standard fashion. Then our logic $\mathbb{L}$ is given by the following sequent calculus, where proof trees are introduced by axioms (this is simply a reprinting of Figure 3 above):[24]

Axiom 1: If $\Gamma \vdash_0 \Theta$, then $\Gamma \vdash \Theta$ may form the base of a proof tree.

$$\frac{\Gamma \vdash \Theta, A \qquad B, \Gamma \vdash \Theta}{A \to B, \Gamma \vdash \Theta}\,\text{L}\to \qquad\qquad \frac{A, \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \to B}\,\text{R}\to$$

$$\frac{\Gamma, A, B \vdash \Theta}{\Gamma, A\&B \vdash \Theta}\,\text{L\&} \qquad\qquad \frac{\Gamma \vdash A, \Theta \qquad \Gamma \vdash B, \Theta}{\Gamma \vdash A\&B, \Theta}\,\text{R\&}$$

$$\frac{A, \Gamma \vdash \Theta \qquad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta}\,\text{L}\vee \qquad\qquad \frac{\Gamma \vdash A, B, \Theta}{\Gamma \vdash A \vee B, \Theta}\,\text{R}\vee$$

$$\frac{\Gamma \vdash A, \Theta}{\neg A, \Gamma \vdash \Theta}\,\text{L}\neg \qquad\qquad \frac{A, \Gamma \vdash \Theta}{\Gamma \vdash \neg A, \Theta}\,\text{R}\neg$$

Note that $\vdash_0$ and $\vdash$ here relate multisets. I treat things in this manner in order to avoid assuming *any* structural features absent permutation. I call $\mathbb{L}$ here NM-MS since its consequence relation is given by a <u>N</u>on-<u>M</u>onotonic <u>M</u>ulti-<u>S</u>uccedent sequent calculus.

Next I rehearse some important results for NM-MS .[25]

Theorem 2.2.4. *If $\Gamma \vdash \Theta$ may be arbitrarily weakened with atoms, then it may be arbitrarily weakened with logically complex sentences:*

$$\forall \Delta_0, \Lambda_0 \subseteq \mathcal{L}_0(\Delta_0, \Gamma \mid\!\sim \Theta, \Lambda_0) \Leftrightarrow \forall \Delta, \Lambda \subseteq \mathcal{L}(\Delta, \Gamma \mid\!\sim \Theta, \Lambda).$$

[24]The rules are the same as Ketonen uses. The rules with two top sequents are additive; the rules with a single top-sequent are multiplicative. These are sometimes called "mixed" or "assorted" rules/connectives (see e.g. Dicher, 2016). It is similar to the system called G3cp discussed in (Negri et al., 2008, ch. 3) with a more standard treatment of negation and material axioms. As is well known, these rules are equivalent to the multiplicative and additive rules of linear logic given monotonicity and contraction (Girard, 2011).

[25]Note that many of these results have full proofs worked out in (Girard, 2011; Negri et al., 2008). Since my system is slightly different than the systems featured there, a more thorough treatment would require some minor modification.

Proof. ($\Leftarrow$) is immediate. ($\Rightarrow$) is proven by induction on the complexity of $\Delta \cup \Lambda$ where complexity is understood in terms of the complexity of the most complex sentences in $\Delta \cup \Lambda$. ∎

A similar result is in the offing, namely that the sequent calculus preserves contraction.

Theorem 2.2.5. *If* $\Gamma \mid\!\sim \Theta$ *allows contraction of atomic sentences, then it allows contraction of logically complex sentences.*

Proof. One direction is trivial, the other direction is provided by induction on the complexity of the contracted sentence. ∎

Since it is well known that the rules featured above are equivalent to both the additive and multiplicative rules of linear logic given contraction and monotonicity, we can actually locate the condition needed for our logic to be supra-classical.

Definition 2.2.6. We say that $\vdash_0$ obeys Containment (CO) if

$$\forall \Delta, \Lambda \subseteq \mathcal{L}_0(\Delta, p \vdash_0 p, \Lambda)$$

(i.e. if we have $\forall q \in \mathcal{L}_0(q \vdash_0 q)$ and all such sequents may be arbitrarily weakened; the fragment carved out by this stipulation will also obviously obey contraction). In short: let us define $\vdash_0^{CO}$ such that $\vdash_0^{CO}$ obeys reflexivity $\forall q \in \mathcal{L}_0(q \vdash_0 q)$, weakening and contraction. And further stipulate that no proper subset of $\vdash_0^{CO}$ obeys all of these conditions. A base consequence relation $\vdash_0$ is said to obey CO iff it includes $\vdash_0^{CO}$, i.e. $\vdash_0^{CO} \subseteq \vdash_0$.

Theorem 2.2.7. *If* $\vdash_0$ *obeys CO, then* $\vdash$ *is supra-classical.*

Proof. Result is well known, but can be easily established by showing an equivalence with Gentzen's LK in the fragment of $\vdash$ generated by $\vdash_0^{CO}$. ∎

Finally, the next theorem is of particular import to the sections following this one.

Theorem 2.2.8. *All rules of the sequent calculus are reversible. That is, if* $\Gamma \vdash \Theta$ *would be the result of the application of a rule to* $\Gamma^* \vdash \Theta^*$ *(and possibly* $\Gamma^{**} \vdash \Theta^{**}$*) then*

$$\Gamma \vdash \Theta \Leftrightarrow \Gamma^* \vdash \Theta^* (and \quad \Gamma^{**} \vdash \Theta^{**}).$$

Proof. Proof is straightforward by induction on proof height. ∎

From this my first gloss on logical expression follows immediately. In the next section I prove that the more precise sense (in Definition 2.2.1) also holds.

Corollary 2.2.9. *The sequent calculus is conservative. That is*

$$\Gamma \vdash_0 \Theta \Leftrightarrow \Gamma \vdash \Theta.$$

Next I show how consequence relations may be represented in NM-MS. First two central results concerning conjunctive and disjunctive normal forms.[26]

Proposition 2.2.10. Let $CNF(A)$ be the conjunctive normal form representation of A. It follows that

$$\Gamma \vdash \Theta, A \Leftrightarrow \Gamma \vdash \Theta, CNF(A).$$

Proof. Proof proceeds constructively. From theorem 2.2.8, we may deconstruct A until we have a number of sequents of the form: $\Gamma \vdash \Theta, \Lambda_1$; $\Gamma \vdash \Theta, \Lambda_2; \ldots$ $\Gamma \vdash \Theta, \Lambda_n$ where $\Lambda_i (1 \leq i \leq n)$ contains only literals. We next construct $CNF(A)$ via repeated application of R$\vee$ and R&:

$$\Gamma \vdash \Theta, (\bigvee \Lambda_1) \& (\bigvee \Lambda_2) \& \cdots \& (\bigvee \Lambda_n),$$

i.e. $\Gamma \vdash \Theta, CNF(A)$. ∎

Proposition 2.2.11. Let $DNF(A)$ be the disjunctive normal form representation of A. It follows that

$$A, \Gamma \vdash \Theta \Leftrightarrow DNF(A), \Gamma \vdash \Theta.$$

Proof. Proof is identical to the previous proposition except the sets are on the left and we construct $DNF(A)$ via L& and L$\vee$. ∎

Theorem 2.2.12 (Representation Theorem 1)**.** *Let CR be a consequence relation, i.e. $CR \subseteq \mathcal{P}(\mathcal{L})^2$. Then we may specify what must be included in $\vdash_0$ such that $CR \subseteq \vdash$.*

[26]Note that the results in Propositions 2.2.10 and 2.2.11 follow closely the distribution properties Girard demonstrates for different connectives in linear logic (Girard, 1987, 2011).

Proof. Proof proceeds constructively. For each $\Gamma \vdash \Theta$ in CR let us find an equivalent $CNF(A) \vdash CNF(B)$. This has the form:

$$(\&\Gamma_1) \vee \cdots \vee (\&\Gamma_a) \vdash (\bigvee\Theta_1)\& \cdots \&(\bigvee\Theta_b).$$

This holds just in case (for $1 \leq i \leq a$ and $1 \leq j \leq b$) $\Gamma_i \vdash_0 \Theta_j$. Thus we stipulate of the base that $\Gamma_i \vdash_0 \Theta_j$ for $1 \leq i \leq a$ and $1 \leq j \leq b$. If we do this for each implication in CR then we are guaranteed that $CR \subseteq \vdash$. ■

Theorem 2.2.13 (Representation Theorem 2)**.** *Let CR be a consequence relation. If CR is closed under some modest syntactic constraints,*[27] *then we may specify $\vdash_0$ such that $CR = \vdash$.*

Proof. Proof is identical to the first Representation Theorem except that the syntactic constraints on CR have it that $\mid\!\sim = CR$. ■

These results give us a way of saying exactly how to reconstruct arbitrary consequence relations using my machinery and given some modest constraints how to reconstruct them *exactly*. It is this ability to reconstruct consequence relations *exactly* that will prove most important. For what it shows is that we are able to find exactly which pre-logical implications an arbitrary implication involving logical vocabulary *expresses*. That is, what I have shown is a method for finding exactly which implications in $\vdash_0$ are expressed by each implication in $\vdash$. We are thus in a position to prove the following straight away.

Theorem 2.2.14 (Expressivity)**.** *NM-MS is expressive. That is, we have*

$$\Gamma \mid\!\sim \Theta \Leftrightarrow (\Gamma_1 \mid\!\sim_0 \Theta_1 \bigwedge \cdots \bigwedge \Gamma_n \mid\!\sim_0 \Theta_n).$$

for some $\Gamma_1, \Theta_1, \ldots, \Gamma_n, \Theta_n$ and arbitrary $\mid\!\sim_0$.

[27] In a more formal account I treat representation as of *theories* (see the appendix). Here I characterize it in terms of consequence relations, where we are able to precisely represent a consequence relation just in case it is closed under the rules of NM-MS. In the case where we wish to treat theories instead, then a theory T must meet the following constraints: &-composition and -decomposition ($A, B \in T$ iff $A\&B \in T$), Distribution (of $\vee$ over &) $(A \vee (B\&C)) \in T$ iff $(A \vee B)\&(A \vee C) \in T$, Conditional Equivalence ($A \to B = \sigma$ is a sub-formula of $\tau \in T$ iff $\neg A \vee B = \sigma'$ is a subformula of $\tau \in T$), both De-Morgan's Equivalences (likewise defined over sub-formulae) and involution (also defined over subformulae).

Proof. Suppose $\Gamma \vdash \Theta$ and let it be equivalent to $DNF(A) \vdash CNF(B)$ for some A and B. This has the form:

$$(\&\Gamma_1) \vee \cdots \vee (\&\Gamma_a) \vdash (\bigvee\Theta_1)\& \cdots \&(\bigvee\Theta_b).$$

This holds just in case (for $1 \leq i \leq a$ and $1 \leq j \leq b$) $\Gamma_i \vdash_0 \Theta_j$. Next, let us enumerate $\langle i, j\rangle$ as $1, \ldots, n$. Then we have that:

$$\Gamma \vdash \Theta \Leftrightarrow (\Gamma_1 \vdash_0 \Theta_1 \bigwedge \cdots \bigwedge \Gamma_n \vdash_0 \Theta_n).$$

■

2.2.3 Recovering Structure: Sf-Expression

I have so far shown how NM-MS is expressive in the sense made precise in Definition 2.2.1. Now I show how NM-MS may express particular structural features. First I introduce a schema for introducing a piece of logical vocabulary '$\boxed{\mathfrak{S}}$'.

First, let us enrich our sequent calculus by introducing a second turnstile $\vdash^{\mathfrak{S}}$. Now let $\vdash_0^{\mathfrak{S}}$ pick out some subset of $\vdash_0$. Later I will discuss principles for determining which subset, but for now I leave the details vague. We may introduce the following rules to our sequent calculus:[28]

Axiom 2: If $\Gamma \vdash_0^{\mathfrak{S}} \Theta$ then $\Gamma \vdash^{\mathfrak{S}} \Theta$.

$$\frac{A, \Gamma \vdash^{\mathfrak{S}} \Theta}{\boxed{\mathfrak{S}}A, \Gamma \vdash^{[\mathfrak{S}]} \Theta}\text{L}\boxed{\mathfrak{S}} \qquad\qquad \frac{\Gamma \vdash^{\mathfrak{S}} \Theta, A}{\Gamma \vdash^{[\mathfrak{S}]} \Theta, \boxed{\mathfrak{S}}A}\text{R}\boxed{\mathfrak{S}}$$

Lemma 2.2.15. *L$\boxed{\mathfrak{S}}$ and R$\boxed{\mathfrak{S}}$ are reversible rules.*

We thus have the following result.

Theorem 2.2.16. *Let $\mathfrak{Sf}$ be a structural rule. Suppose that $\mathfrak{Sf}$ is* preserved *(in the sense of Definition 2.2.2) and suppose further that $\vdash^{\mathfrak{S}}$ marks that structural feature exactly. We*

[28]Note that the rest of our sequent calculus is altered such that our other rules preserve $\vdash^{\mathfrak{S}}$. E.g. R& requires that both top sequents have either $\vdash$ or $\vdash^{\mathfrak{S}}$ (I do not allow mixed cases).

thus have: $\mathfrak{Sf}(\Gamma \vdash \Theta)$ *iff* $\Gamma \vdash^{\mathfrak{S}} \Theta$. *It follows that* $\boxed{\mathfrak{S}}$ expresses *(in the sense of Definition 2.2.3)* $\mathfrak{Sf}$. *Thus:*

$$\boxed{\mathfrak{S}}A, \Gamma \vdash \Theta \Leftrightarrow \mathfrak{Sf}(A, \Gamma \vdash \Theta)$$

$$\Gamma \vdash \Theta, \boxed{\mathfrak{S}}A \Leftrightarrow \mathfrak{Sf}(A, \Gamma \vdash \Theta, A)$$

Proof. I prove only the latter biconditional since the proof of the former is identical. By supposition $\mathfrak{Sf}(\Gamma \vdash \Theta, A)$ iff $\Gamma \vdash^{\mathfrak{S}} \Theta, A$. Since it follows that our R$\boxed{\mathfrak{S}}$ rule is reversible, we have that $\Gamma \vdash^{\mathfrak{S}} \Theta, A$ iff $\Gamma \vdash \Theta, \boxed{\mathfrak{S}}A$. Thus

$$\Gamma \vdash \Theta, \boxed{\mathfrak{S}}A \Leftrightarrow \mathfrak{Sf}(\Gamma \vdash \Theta, A).$$

■

The result of the above proof is a general method for introducing logical vocabulary that is *expressive* of structural features. If the rules for the logical vocabulary's introduction are reversible and the structural feature in question is *preserved* by $\mathbb{L}$, then the logical vocabulary will *express* that structural feature. I next rehearse two specific cases of this: an operator that marks monotonicity and an operator that marks classical validity.

2.2.3.1 Expressing Structural Features

The rules for monotonicity have the following form:

Axiom 2: If $\forall \Delta, \Lambda \subseteq \mathcal{L}_0(\Delta, \Gamma \vdash_0 \Theta, \Lambda)$ then $\Gamma \vdash^M \Theta$.

$$\frac{A, \Gamma \vdash^M \Theta}{\boxed{M}A, \Gamma \vdash^{[M]} \Theta}\ \mathrm{L}\boxed{M} \qquad\qquad \frac{\Gamma \vdash^M \Theta, A}{\Gamma \vdash^{[M]} \Theta, \boxed{M}A}\ \mathrm{R}\boxed{M}$$

I have already shown in Theorem 2.2.4 that weakening is preserved by the rules of NM-MS. It therefore follows that:

Corollary 2.2.17. $\boxed{M}$ *expresses weakening/monotonicity. That is,*

$$\boxed{M}A, \Gamma \vdash \Theta \Leftrightarrow \forall \Delta, \Lambda(\Delta, A, \Gamma \vdash \Theta, \Lambda)$$

$$\Gamma \vdash \Theta, \boxed{M}A \Leftrightarrow \forall \Delta, \Lambda(\Delta, \Gamma \vdash \Theta, A, \Lambda)$$

This means that we may expand NM-MS (our $\mathbb{L}$) in order to mark *in the object language* which implications are persistent under arbitrary weakenings. Next, I show a similar result for contraction. That is, I show a way of marking sequents that are where contraction holds. We introduce the following axiom and rules as before:

Axiom 2: If $\Gamma \vdash_0 \Delta$ and for arbitrary Δ, Λ we have $\Delta \vdash_0 \Lambda$ if this would be the result of some number of applications of contraction to $\Gamma \vdash_0 \Delta$, then $\Gamma \vdash^C \Theta$.

$$\frac{A, \Gamma \vdash^C \Theta}{\boxed{C}A, \Gamma \vdash^{[C]} \Theta}\,\mathrm{L}\boxed{C} \qquad\qquad \frac{\Gamma \vdash^C \Theta, A}{\Gamma \vdash^{[C]} \Theta, \boxed{C}A}\,\mathrm{R}\boxed{C}$$

I have already shown in Theorem 2.2.5 that contraction is preserved by the rules of NM-MS. It therefore follows that:

Corollary 2.2.18. $\boxed{C}$ *expresses contraction. That is,*

Next, I demonstrate the same for "classicality", i.e. develop an operator that marks implications that are valid classically.

Axiom 2: If $\Gamma, p \vdash_0 p, \Theta$ then $\Gamma, p \vdash^K p, \Theta$ (where Γ, Θ may be possibly empty).

$$\frac{A, \Gamma \vdash^K \Theta}{\boxed{K}A, \Gamma \vdash^{[K]} \Theta}\,\mathrm{L}\boxed{K} \qquad\qquad \frac{\Gamma \vdash^K \Theta, A}{\Gamma \vdash^{[K]} \Theta, \boxed{K}A}\,\mathrm{R}\boxed{K}$$

Again, I have already shown in Theorem 2.2.7 that classicality is a feature NM-MS preserves. Thus any sequent which is derived from atomic sequents which are part of the CO (cf. Definition 2.2.6) fragment of $\vdash_0$ (regardless of whether $\vdash_0$ actually obeys CO) will be classically valid.

Corollary 2.2.19. *Let* $\vdash_{LK}$ *be the consequence relation instantiated by Gentzen's LK minus the rules for quantifiers (and with* $\wedge$ *substituted with &, etc.). Then* $\boxed{K}$ *expresses classical validity, that is:*

$$\boxed{K}A, \Gamma \vdash \Theta \Leftrightarrow A, \Gamma \vdash_{LK} \Theta$$

$$\Gamma \vdash \Theta, \boxed{K}A \Leftrightarrow \Gamma \vdash_{LK} \Theta, A$$

There are of course many further possibilities for such '$\boxed{\mathfrak{S}}$' operators. We may also introduce vocabulary for expressing inference that obey, transitivity + weakening, more restricted weakening principles, and perhaps more.[29]

I will introduce one more such notion. I introduce it because it will have an interesting philosophical usage in the fourth chapter of the dissertation (though I only make short reference to it there). That is a way of marking regions of the consequence relation which are:

- Supra-classical (contain CO)
- Monotonic
- Transitive
- Contractive

I claim that this combination of features captures intuitively the notion of a sentence's "literal meaning". What follows form a sentence, strictly speaking (or taken literally),[30] should follow from that sentence regardless of the context under which it is being considered (or what one uses the sentence to mean). Included within this are all of the classical consequences as well as consequences which are persistent under arbitrary weakenings. It also seems to me that sentences, taken literally, should contract (since we're isolating a fragment of the sentence meant to express something like a "minimal proposition").[31]

We introduce the following rules for "literally".

Axiom 2: It is simpler to define the anti-extension. Suppose $A, \Gamma \vdash_0 \Theta$. Then $A, \Gamma \not\vdash^L \Theta$ if any of the following hold:

- Exists $p \in \{A\} \cup \Gamma \cup \Theta$ and $\langle \Lambda, \Delta \rangle$ such that $\Lambda, p \not\vdash_0 p, \Delta$.
- $\Gamma \vdash_0 \Theta, A$ but $\Gamma \not\vdash_0 \Theta$
- $\Delta \not\vdash_0 \Theta$ where this would be the result of contracting the original sequent
- There exists $\langle \Lambda, \Delta \rangle$ such that $A, \Lambda, \Gamma \not\vdash_0 \Theta, \Delta$.

[29] Makinson for example considers a consequence relation which is supra-classical, monotonic, and obeys transitivity (Makinson, 1994, 2005). We could introduce an operator to express exactly this consequence relation along with some other consequence relations discussed therein.

[30] The locution "what follows strictly speaking" shouldn't be read to mean "strict implication". That is *not* what I mean to capture.

[31] The significance of contraction here might not be fully appreciated until the next chapter.

Similarly for $\Gamma \vdash_0 \Theta, A$.

$$\frac{A, \Gamma \vdash^{L} \Theta}{\boxed{L} A, \Gamma \vdash^{[L]} \Theta}\ \mathrm{L}\boxed{L} \qquad\qquad \frac{\Gamma \vdash^{L} \Theta, A}{\Gamma \vdash^{[L]} \Theta, \boxed{L} A}\ \mathrm{R}\boxed{L}$$

NM-MS already preserves contraction, weakening, and classicality. So we must show that transitivity is obeyed in that fragment as well. We show the result we want directly.

Theorem 2.2.20. *The following rule would be eliminable:*[32]

$$\frac{A, \Gamma \vdash^{L} \Theta \qquad \Gamma \vdash^{L} \Theta, A}{\Gamma \vdash^{L} \Theta}$$

Proof. It is sufficient to show we may push cut up the proof tree via induction on complexity of A. The base case is immediate via Axiom 2 and the reversibility of the rules. Here is how the inductive step works for conjunction: suppose A is of the form $A\&B$. Then the top sequents are:

$$A\&B, \Gamma \vdash^{L} \Theta,$$
$$\Gamma \vdash^{L} \Theta, A\&B.$$

Since the rules are reversible we have:

$$A, B, \Gamma \vdash^{L} \Theta,$$
$$\Gamma \vdash^{L} \Theta, A,$$
$$\Gamma \vdash^{L} \Theta, B.$$

Since monotonicity is preserved we are guaranteed: $B, \Gamma \vdash^{L} \Theta, A$. Via the inductive hypothesis we have:

$$B, \Gamma \vdash^{L} \Theta.$$

Via another invocation of the same we have our result:

$$\Gamma \vdash^{L} \Theta.$$

■

[32] Because this fragment is both monotonic and contractive I use a shared-context version of cut. We could just as well have used the rule:

$$\frac{A, \Gamma \vdash^{L} \Theta \qquad \Lambda \vdash^{L} \Delta, A}{\Gamma, \Lambda \vdash^{L} \Theta, \Delta}$$

Thus we have a way of marking the confluence of all of these structural features *in the object language*. I claim that this *expresses* "minimal propositions" (or the literal meaning) of a sentence. This does not mean that I endorse these ideas. But I do think that for sentences that seem to have them, that we can mark this behavior.[33]

2.2.3.2 Some Defective Cases

So far I have introduced a more precise criterion for understanding logical expressivism and in particular for understanding how *structural* features of inference might be expressed. I then introduced a system that was not only *expressive* in this sense, but also successfully preserved and expressed several important structural features. In order to appreciate exactly what I am up to, however, it will be useful to look at some cases where each of these criteria fail.

Example 2.2.21. The multiplicative rules of linear logic are *not expressive*. I show that this is the case for the multiplicative conjunction $\otimes$:

$$\dfrac{\Gamma, A, B \vdash \Theta}{\Gamma, A \otimes B \vdash \Theta}\,\mathrm{L}\otimes \qquad\qquad \dfrac{\Gamma \vdash \Theta, A \qquad \Delta \vdash \Lambda, B}{\Gamma, \Delta \vdash \Theta, \Lambda, A \otimes B}\,\mathrm{R}\otimes$$

It is sufficient to show a case where the logic does not express particular implications in $\vdash_0$. Notice that there are potentially two ways to derive $p \otimes q \vdash p \otimes q$ where $p, q \in \mathcal{L}_0$:

$$\dfrac{\dfrac{p \vdash p \qquad q \vdash q}{p, q \vdash p \otimes q}\,\mathrm{R}\otimes}{p \otimes q \vdash p \otimes q}\,\mathrm{L}\otimes \qquad\qquad \dfrac{\dfrac{p, q \vdash q}{p \otimes q \vdash q}\,\mathrm{L}\otimes \qquad \vdash p}{p \otimes q \vdash p \otimes q}\,\mathrm{R}\otimes$$

Since the atomic sequents used to start each proof tree are different (in fact they are entirely different), it's possible that $\vdash_0$ includes one and $\vdash_0'$ includes the other and thus the presence of $p \otimes q \vdash p \otimes q$ does not guarantee the presence of either. In this sense, logics which include '$\otimes$' are not expressive in the relevant sense.

It is also possible to find counter-examples to Sf-Preservation and Sf-Expression. Even using the rules of NM-MS such counter-examples will arise:

[33] Hopefully these remarks aren't too opaque. Proponents who endorse the idea of a literal meaning often take it that the assertion of "p" and the assertion of "p literally" (where literally is appropriately inserted) are identical. I certainly don't claim that $\boxed{L}p$ and p are the same.

Example 2.2.22. Suppose we want to introduce an operator '$\boxed{R}$' to mark instances of reflexivity, i.e. $\phi \vdash \phi$. Then the rules for introducing such an operator should probably have the form:

Axiom 2: If $p \vdash_0 p$ then $p \vdash^R p$.

$$\frac{A, \Gamma \vdash^R \Theta}{\boxed{R}A, \Gamma \vdash^{[R]} \Theta}\ \mathrm{L}\boxed{R} \qquad\qquad \frac{\Gamma \vdash^R \Theta, A}{\Gamma \vdash^{[R]} \Theta, \boxed{R}A}\ \mathrm{R}\boxed{R}$$

Unfortunately, it is easy to show that NM-MS fails to preserve reflexivity and thus fails to express it. For example $A\&B \vdash A\&B$ is clearly an instance of reflexivity and thus we should want $A\&B \vdash \boxed{R}(A\&B)$. But clearly $A\&B \vdash A\&B$ must be derived from $A, B \vdash A$ and $A, B \vdash B$, neither of which are instances of reflexivity.[34]

There will therefore be logics which in general fail to be expressive and even among those that are expressive there will be structural features that fail to be preserved and thus expressed. Deciding how expressive one wants one's logic to be and which structural features ought to be preserved are therefore *not* independent questions.

2.3 Why go substructural?

So far I have motivated a formal semantics based upon the idea that (i) the content of a sentence should be understood in terms of the contribution that that sentence makes to good implication and (ii) that such implications may be radically substructural (i.e. may disobey monotonicity, transitivity, contraction, and/or reflexivity in potentially unpredictable ways). I have also shown that despite the radically substructural nature of the implication relation that underlies this notion of content, that a perfectly tractable semantics emerges on which much work can be done, including a proof of a representation theorem. In the next chapter,

[34]Though they are both found in the region of the consequence relation which allows reflexivity *together with weakening*, hence why we are able to have an operator to mark classicality.

It is also worth remarking that the above might also fail for independent reasons. For example, if we are able to derive $A\&B \vdash A\&B$, then we could also derive $A\&B \vdash B\&A$, but is the latter here an instance of the structural feature of reflexivity? It is not obvious that we should think so. In general, even when a sequent calculus preserves reflexivity, it needn't generate *only* reflexive sequents from the reflexive fragment of its axioms.

I will try to get clearer on what makes this approach to semantics particularly compelling and why it has been so far under-explored (i.e. I intend to diagnose what has motivated an avoidance of this approach). Before I can do that, however, I want to put forward some brief arguments for why we should even go substructural (as opposed to accounting for this phenomena in some other manner).

The argument in this section are therefore meant to motivate and not decide the matter. In short: I've shown that you *can* construct such a semantics, now I want to show why you might *want to* (and next: why motivations against such a project are misguided). To do this, I'll examine one way of trying to account for substructural consequence that fails to bear fruit. My arguments here are meant to be philosophical, not technical. In a fuller treatment, I believe that similar arguments could be made for other structural features, but I only pursue one such argument here. In particular, I argue that attempts to understand non-monotonicity in terms of defeat or defeasible reasoning flounder. This is because the central notion required for such accounts—that of a "defeater"—cannot be made sense of. Instead, I hope to illustrate that when Γ implies A but some superset of Γ doesn't:

$$\Gamma \vdash A$$

$$\Gamma, \Delta \nvdash A.$$

It need not be because of some defeater in the latter case, but rather because of the way in which the various considerations in $\Gamma \cup \Delta$ hang together. To understand this phenomenon in terms of how these considerations hang together is precisely to understand it as a structural phenomenon, i.e. in terms of non-monotonicity rather than defeat.

First, I'll briefly rehearse how defeat has been understood in the epistemology and defeasible reasoning literatures (§2.3.1). Following this, I try to isolate a tractable notion of defeat with ever weaker conceptions of it (§2.3.2). But none of these are shown to work.

2.3.1 Understanding Defeaters: An Exercise in Defeat

The concept of defeat has figured prominently within epistemology since Gettier (1963). The counter-examples that Gettier employs against the justified true belief (JTB) account

Appendix Unabridged Technical Results

Early in this dissertation I introduce an "implicational phase space semantics". I prove some results concerning this early on and refer back to those results (as well as introduce new results) throughout this dissertation. The results introduced in the dissertation often feature abbreviated proofs (meant to give the shape of a proof without taking up too many pages). The point of this appendix is to collect all of the machinery and proofs together into one location and for the proofs to be presented in an unabridged fashion. Because all of the material appears elsewhere in the dissertation and because I assume that only the most enthusiastic reader will consult this appendix, I am rather sparse on philosophical details here.

Throughout this paper I have introduced in a piecemeal fashion some technical machinery. I have also in a piecemeal fashion and without proper justification claimed certain results concerning this apparatus. In this section I systematically explain how the machinery works independently of its philosophical application above. The following appendices should therefore also serve as a kind of reference for locating features of that system.

A.1 Proof Theory

In this section I rehearse the sequent calculus.

Definition A.1.1 (Base Language and Consequence Relation)**.** A base language $\mathcal{L}_0$ consists of a set of atomic sentence letters: $p_1, \ldots, p_n$.

A base consequence relation relates multi-sets of atoms to multi-sets of atoms:[1]

$$\vdash_0 \subseteq \mathcal{P}(\mathcal{L}_0)^2.$$

We may express such a relation as follows:

$$p_1, \ldots, p_n \vdash_0 q_1, \ldots, q_m,$$

I will typically omit set bracket and union symbols where lack of ambiguity allows. As a convention I will use capital Greek latters $(\Gamma, \Delta, \ldots)$ to represent multi-sets of sentences, capital Latin letters to represent sentences and lowercase Latin letters for atomic sentences (i.e. sentences contained in $\mathcal{L}_0$). I next introduce several important conditions base consequence relations may satisfy.

Definition A.1.2 (Conditions on Base Consequence Relations)**.** The following conditions are of special interest:

Reflexivity A base consequence relation is said to be **reflexive** iff

$$\forall p \in \mathcal{L}_0(p \vdash_0 p).$$

Monotonicity A base consequence relation is said to be **monotonic** iff[2]

$$\Gamma \vdash_0 \Theta \Rightarrow \forall \Delta, \Lambda \subseteq \mathcal{L}_0(\Gamma, \Delta \vdash_0 \Theta, \Lambda).$$

[1] Note that the object language consists of "multi-sets", i.e. sets distinguished by multiplicity. While $\{a\}$ and $\{a, a\}$ are the same set, they are *not* the same *multi*-set. For the most part, I don't rely on anything too controversial surrounding multi-sets, but this allows for less ambiguity surrounding how the sequent calculus works.

Note in particular, that I sometimes use notation that is technically ambiguous. For example, I below write that $\vdash_0 \subseteq \mathcal{P}(\mathcal{L}_0)^2$ even though $\vdash_0$ is a relation between multi-sets and $\mathcal{L}_0$ is a set. We should understand $\mathcal{P}(\mathcal{L}_0)$ in this context as the **set** of every possible finite **subset** of $\mathcal{L}_0$ with every possible multiplicity of its members, so that $\mathcal{P}(\mathcal{L}_0)$ is actually a **set** of **multi-sets**, and $\mathcal{P}(\mathcal{L}_0)^2$ is a **set** of ordered pairs of **multi-sets**. In general context shall settle any possible ambiguity here.

[2] Note that I use '$\Rightarrow$', '$\Leftarrow$', and '$\Leftrightarrow$' to signal "meta-theoretic entailment/conditionals". That is $A \Rightarrow B$ means that if A is the case, then B must be the case (via ordinary mathematical/logical reasoning in the meta-theory). I include this note to avoid any potential confusion regarding notation.

Contractivity A base consequence relation is said to be **contractive** iff

$$p, p, \Gamma \vdash_0 \Theta \Rightarrow p, \Gamma \vdash_0 \Theta,$$

and

$$\Gamma \vdash_0 \Theta, p, p \Rightarrow \Gamma \vdash_0 \Theta, p.$$

Flatness A base consequence relation is said to be **flat** iff there is a subset of $\vdash_0$ that is reflexive, monotonic, and contractive. That is, iff

$$\forall p \in \mathcal{L}_0 \forall \Delta, \Lambda \subseteq \mathcal{L}_0 (\Delta, p \vdash_0 p, \Lambda).$$

Minimal, Flat $\vdash_0 \subseteq \mathcal{P}(\mathcal{L}_0)^2$ is said to be a **minimal, flat** base consequence relation iff $\vdash_0$ is flat and no proper subset of $\vdash_0$ is also flat.

A.1.1 The Extension

The base language is extended in the standard fashion to include $\{\neg, \vee, \&, \rightarrow\}$:

- If $p \in \mathcal{L}_0$ then $p \in \mathcal{L}$. I call all such 'p' atoms.
- If $A, B \in \mathcal{L}$ then $\ulcorner A \rightarrow B \urcorner, \ulcorner A \& B \urcorner, \ulcorner A \vee B \urcorner \in \mathcal{L}$.
- If $A \in \mathcal{L}$, then $\ulcorner \neg A \urcorner \in \mathcal{L}$.

The base consequence relation ($\vdash_0$) is extended to $\vdash \subseteq \mathcal{P}(\mathcal{L})^2$ according to the following sequent calculus. The axioms, which are the only sequents which may serve as roots of our proof trees are generated as follows:

Axiom If $\Gamma \vdash_0 \Theta$, then $\Gamma \vdash \Theta$.

The rules follow below. I use the Ketonen rules for sentential logic, also sometimes called "assorted" rules (since they mix additive and multiplicative connectives) in the sequent calculus.[3] $\Gamma \vdash \Theta$ is included in $\vdash$ iff there is a deduction from the axioms to $\Gamma \vdash \Theta$ in accordance with these rules:

[3] See Ketonen (1944). See Dicher (2016) for this usage of "assorted" (and also confer Humberstone (2007)). In the Linear Logic, these are sometimes also called "asynchronous" (as opposed to "synchronous") rules. See Andreoli (1992); Liang and Miller (2009). The rules and sequent calculus is also extremely similar to e.g. G3cp, see Negri et al. (2008).

$$\frac{\Gamma \vdash \Theta, A \qquad B, \Gamma \vdash \Theta}{A \rightarrow B, \Gamma \vdash \Theta}\text{L}\rightarrow \qquad \frac{A, \Gamma \vdash \Theta, B}{\Gamma \vdash A \rightarrow B, \Theta}\text{R}\rightarrow$$

$$\frac{\Gamma, A, B \vdash \Theta}{\Gamma, A\&B \vdash \Theta}\text{L\&} \qquad \frac{\Gamma \vdash \Theta, A \qquad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A\&B}\text{R\&}$$

$$\frac{A, \Gamma \vdash \Theta \qquad B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta}\text{L}\vee \qquad \frac{\Gamma \vdash \Theta, A, B}{\Gamma \vdash \Theta, A \vee B}\text{R}\vee$$

$$\frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta}\text{L}\neg \qquad \frac{A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta, \neg A}\text{R}\neg$$

A.1.2 Some Central Results

First a well known result that will be extremely useful.

Theorem A.1.3. *All of the rules of the sequent calculus are reversible. That is, if* $\Gamma \vdash \Theta$ *would be the result of a rule application to* $\Gamma^* \vdash \Theta^*$ *(and possibly* $\Gamma^{**} \vdash \Theta^{**}$*):*

$$\frac{\Gamma^* \vdash \Theta^* \qquad [\Gamma^{**} \vdash \Theta^{**}]}{\Gamma \vdash \Theta} *$$

It follows that

$$\Gamma \vdash \Theta \Leftrightarrow \Gamma^* \vdash \Theta^* (\textit{and possibly } \Gamma^{**} \vdash \Theta^{**}).$$

Proof. By induction on proof height. If $\Gamma \vdash \Theta$ is the result of a proof of height 1 (i.e. only one rule application) then the result is immediate. Therefore suppose the result holds for proof trees of height less than n and suppose $\Gamma \vdash \Theta$ is the result of a proof tree of height n. We must show that the result holds regardless of which rule '*' is. I show the result holds when '*' is L→ (from which the cases where '*' is L∨ or R& may be proven in an analogous fashion) and when '*' is R→ (from which the cases where '*' is L&, R∨, L¬, or R¬ may be proven in an analogous fashion.

There are two sub-cases: the last step is either the result of a rule with one-top sequent (R→, L&, R∨, L¬, R¬) or the result of a rule with two-top sequents (L←, L∨, R&). We must show that our result holds for all of our rules.

Case 1.1: Suppose '*' is L→ and the last step is the application of a rule with exactly one top-sequent. Then the last step is of the form:

$$\frac{A \to B, \Gamma' \vdash \Theta'}{A \to B, \Gamma \vdash \Theta}$$

By our inductive hypothesis we have $B, \Gamma' \vdash \Theta'$ and $\Gamma' \vdash \Theta', A$. Via two applications of the same rule we have: $B, \Gamma \vdash \Theta$ and $\Gamma \vdash \Theta, A$.

Case 1.2: Suppose '*' is L$\to$ and the last step is the application of a rule with two top-sequents. If the last step is an application of L$\to$ to our principal formula, then we are done, so suppose this is not the case, then the last step is of the form:

$$\frac{A \to B, \Gamma' \vdash \Theta' \qquad A \to B, \Gamma'' \vdash \Theta''}{A \to B, \Gamma \vdash \Theta}$$

By our inductive hypothesis, we therefore have $B, \Gamma' \vdash \Theta'$ and $\Gamma' \vdash \Theta', A$ as well as $B, \Gamma'' \vdash \Theta''$ and $\Gamma'' \vdash \Theta'', A$. By two applications of the same rule we have $B, \Gamma \vdash \Theta$ and $\Gamma \vdash \Theta, A$.

Case 2.1: Suppose '*' is R$\to$ and the last step is the application of a rule with exactly one top-sequent. If the last step is an application of R$\to$ to our principal formula, then we are sone, so suppose this is not the case, then the last step is of the form:

$$\frac{\Gamma' \vdash \Theta', A \to B}{\Gamma \vdash \Theta, A \to B}$$

By our inductive hypothesis we have $A, \Gamma' \vdash \Theta', B$ and by an application of the same rule we have $A, \Gamma \vdash \Theta, B$.

Case 2.2: Suppose '*' is R$\to$ and the last step is the application of a rule with two top-sequents. Then the last step is of the form:

$$\frac{\Gamma' \vdash \Theta', A \to B \qquad \Gamma'' \vdash \Theta'', A \to B}{\Gamma \vdash \Theta, A \to B}$$

By our inductive hypothesis we have $A, \Gamma' \vdash \Theta', B$ and $A, \Gamma'' \vdash \Theta'', B$. Via an application of the same rule we obtain $A, \Gamma \vdash \Theta, B$.

Cases 3.1,2–8.1,2 Follow in a similar fashion (i.e. when '*' is any of the other rules).

■

The following theorem is also of importance.

Theorem A.1.4. *For every sentence A there exists $A_0^P \subseteq \mathcal{P}(\mathcal{L}_0)^2$ such that*

$$(\forall \langle \Delta, \Lambda \rangle \in A_0^P) A, \Gamma \vdash \Theta \Leftrightarrow \Delta, \Gamma \vdash \Theta, \Lambda.$$

Likewise for every sentence A there exists $A_0^C \subseteq \mathcal{P}(\mathcal{L}_0)^2$ such that

$$(\forall \langle \Delta, \Lambda \rangle \in A_0^P)\Gamma \vdash \Theta, A \Leftrightarrow \Delta, \Gamma \vdash \Theta, \Lambda.$$

These sets (of multi-sets) can be thought of as atomic representatives of A (hence the subscripted 0; the P and C represent whether A appears in the premise/conclusion of an implication).

Proof. I prove both halves of the theorem at once by induction on the complexity of A. That is, I show simultaneously the existence of A_0^P and A_0^C.

Suppose A is atomic, then $A_0^P = \{\langle A, \emptyset \rangle\}$ and $A_0^C = \{\langle \emptyset, A \rangle\}$.

Now suppose our result holds for sentences of complexity strictly less than n and suppose A is of complexity n. A may be a conditional, conjunction, disjunction, or negated sentence.

Suppose A is a conditional, then it is of the form $D \to E$. By hypothesis we have $D_0^P, D_0^C, E_0^P, E_0^C$. Clearly $A_0^P = D_0^C \cap E_0^P$ since we have (from our previous theorem) that $D \to E, \Gamma \vdash \Theta$ iff $E, \Gamma \vdash \Theta$ and $\Gamma \vdash \Theta, D$.

Likewise, $A_0^C = \{\{\Delta \cup \Delta', \Lambda \cup \Lambda'\} | \langle \Delta, \Lambda \rangle \in D_0^P \text{ and } \langle \Delta', \Lambda' \rangle \in E_0^C\}$, since $\Gamma, D \vdash E, \Theta$ iff $\Gamma \vdash D \to E, \Theta$ (so we need the pairwise combination of those sets that comprise D_0^P and E_0^C.

If A is a conjunction or disjunction then we have our result in an analogous manner (consider that $A\&B$ is equivalent to $\neg(A \to B)$ and $A \vee B$ is equivalent to $\neg A \to B$ given our sequent rules).

If A is a negated sentence of the form $\neg B$, then we have by hypothesis B_0^P and B_0^C and so $A_0^P = B_0^C$ and $A_0^C = B_0^P$. ■

We should think of the sets generated in this manner as representing A atomically. That is, they tell us exactly which combination of sets as atoms are needed to build A up in arbitrary contexts (i.e. arbitrary sequents) as either a premise or a conclusion.

From this result, the next result is nearly immediate. First, a definition:

Definition A.1.5 (DNF,CNF)**.** Call a sentence a **literal** if it is either an atom or has the form $\neg A$ where A is an atom.

We say that a sentence A is in **disjunctive normal form (DNF)** if it is a disjunction of conjunctions of literals. I write $DNF(A)$ as shorthand for "the disjunctive normal form of A", where this is understood to mean a sentence that is equivalent to A and in DNF.

We say that a sentence A is in **conjunction normal form (CNF)** if it is a conjunction of disjunctions of literals. I write $CNF(A)$ as shorthand for "the conjunctive normal form of A", where this is understood to mean a sentence that is equivalent to A and in CNF.

Note that in the previous definition and following proposition the expression "equivalent" (sentence) and thus $DNF(A)$ and $CNF(A)$ are ambiguous/not well-defined. The following proposition should therefore be read as establishing the existence of a sentence in DNF that holds exactly when A does, and *mutatis mutandis* for the proposition that follows thereafter.

Proposition A.1.6.

$$A, \Gamma \vdash \Theta \Leftrightarrow DNF(A), \Gamma \vdash \Theta$$

where $DNF(A)$ is the DNF of A.

Proof. Because all rules are reversible, we can simply unfold A in line with Theorem A.1.4 until we have $A_0^P \subseteq \mathcal{P}(\mathcal{L}_0)^2$. Next enumerate A_0^P as $\langle \Delta_1, \Lambda_1 \rangle, \ldots, \langle \Delta_n, \Lambda_n \rangle$. We have

$$A, \Gamma \vdash \Theta \Leftrightarrow \Delta_i, \Gamma \vdash \Theta, \Lambda_i,$$

for $1 \leq i \leq n$. Next let $\neg\Lambda$ be shorthand for $\{\neg\lambda | \lambda \in \Lambda\}$ and $\&\Delta$ be shorthand for $\delta_1 \& \cdots \& \delta_j$ where $\{\delta_1, \ldots, \delta_j\} = \Delta$. We should understand $\bigvee\Delta$ in an analogous fashion. Then we may derive:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Delta_1, \Gamma \vdash \Theta, \Lambda_1}{\vdots}\,\mathrm{L}\neg}{\Delta_1, \neg\Lambda_1, \Gamma \vdash \Theta}\,\mathrm{L}\neg}{\vdots}\,\mathrm{L\&}}{\&(\Delta_1 \cup \neg\Lambda_1), \Gamma \vdash \Theta}\,\mathrm{L\&} \quad \dfrac{\dfrac{\dfrac{\dfrac{\Delta_2, \Gamma \vdash \Theta, \Lambda_2}{\vdots}\,\mathrm{L}\neg}{\Delta_2, \neg\Lambda_2, \Gamma \vdash \Theta}\,\mathrm{L}\neg}{\vdots}\,\mathrm{L\&}}{\&(\Delta_2 \cup \neg\Lambda_2), \Gamma \vdash \Theta}\,\mathrm{L\&}}{\&(\Delta_1 \cup \neg\Lambda_1) \vee \&(\Delta_2 \cup \neg\Lambda_2), \Gamma \vdash \Theta}\,\mathrm{L}\vee \quad \vdots}{\ddots} \quad \dfrac{\vdots \quad \dfrac{\dfrac{\dfrac{\dfrac{\Delta_n, \Gamma \vdash \Theta, \Lambda_n}{\vdots}\,\mathrm{L}\neg}{\Delta_n, \neg\Lambda_n, \Gamma \vdash \Theta}\,\mathrm{L}\neg}{\vdots}\,\mathrm{L\&}}{\&(\Delta_n \cup \neg\Lambda_n), \Gamma \vdash \Theta}\,\mathrm{L\&}}{\cdot^{\cdot^{\cdot}}}}{\&(\Delta_1 \cup \neg\Lambda_1) \vee \cdots \vee \&(\Delta_n \cup \neg\Lambda_n), \Gamma \vdash \Theta}\,\mathrm{L}\vee$$

Clearly

$$\&(\Delta_1 \cup \neg\Lambda_1) \vee \cdots \vee \&(\Delta_n \cup \neg\Lambda_n) = \bigvee\{\&(\Delta_i \cup \neg\Lambda_i) | 1 \leq i \leq n\} = DNF(A).$$

■

We may prove a similar result for the CNF in the succedent.

Proposition A.1.7.

$$\Gamma \vdash \Theta, B \Leftrightarrow \Gamma \vdash \Theta, CNF(B)$$

where $CNF(B)$ is the CNF of B.

Proof. Proof is analogous to proof of previous proposition. We shift all the atomic sets (in i.e. B_0^C) to the right-hand side (using R$\neg$), combine them into disjunctions (using R$\vee$) and finally into conjunctions (using R&) yielding a sentence in CNF. ■

Corollary A.1.8. *For arbitrary* Γ, Θ*, there is some* A *and some* B *such that:*

$$\Gamma \vdash \Theta \Leftrightarrow DNF(A) \vdash CNF(B).$$

Proof. Via repeated applications of L& and R$\vee$ we obtain

$$\&\Gamma \vdash \bigvee\Theta.$$

From the previous two propositions this yields

$$DNF(\&\Gamma) \vdash CNF(\bigvee\Theta).$$

So we are therefore guaranteed the existence of some A (in this case $\&\Gamma$) and some B (in this case $\bigvee\Theta$) such that

$$\Gamma \vdash \Theta \Leftrightarrow DNF(A) \vdash CNF(B).$$

■

Definition A.1.9 (Normal Form)**.** We say that a sequent $\Gamma \vdash \Theta$ is in **Normal Form** if its premise consists of a single sentence in DNF, and its conclusion consists of a single sentence in CNF.

A.1.3 Representation Theorem

Finally I prove my representation theorem. In particular I show constructively that given any theory $\mathcal{T}$ expressible in $\mathcal{L}$ we may specify a base consequence relation $\vdash_0$ such that $\vdash$ realizes $\mathcal{T}$. Further I show that if $\mathcal{T}$ meets certain constraints then we may specify a base consequence relation such that $\vdash$ realizes $\mathcal{T}$ *exactly*.

Definition A.1.10 (Theory)**.** We say that $\mathcal{T}$ is a **theory** of sentential logic if $\mathcal{T} \subseteq \mathcal{L}$. We say that $\vdash$ **realizes** $\mathcal{T}$ iff

$$\forall \tau \in \mathcal{T}(\vdash \tau).$$

We say that $\vdash$ realizes $\mathcal{T}$ **exactly** iff

$$\vdash \tau \Leftrightarrow \tau \in \mathcal{T}.$$

Definition A.1.11 (Inferential Relation)**.** We say that $\mathcal{I}$ is an **inferential relation** if $\mathcal{I} \subseteq \mathcal{P}(\mathcal{L})^2$. We say that $\vdash$ **realizes** $\mathcal{I}$ iff

$$\forall \langle \Gamma, \Theta \rangle \in \mathcal{I}(\Gamma \vdash \Theta).$$

We say that $\vdash$ realizes $\mathcal{I}$ **exactly** iff

$$\Gamma \vdash \Theta \Leftrightarrow \langle \Gamma, \Theta \rangle \in \mathcal{I}.$$

First I show that for any arbitrary theory we may find an inferential relation such that they are always co-realized. Likewise for any arbitrary inferential relation we may find a theory such that they are co-realized. Co-satisfaction here means that the one is realized iff the other is realized. Note that "realization" is always relative to a base-consequence relation.

Proposition A.1.12. Let $\mathcal{T} \subseteq \mathcal{L}$. Then we may find $\mathcal{I} \subseteq \mathcal{P}(\mathcal{L})^2$ such that $\mathcal{I}$ is realized iff $\mathcal{T}$ is realized.

Likewise let $\mathcal{I} \subseteq \mathcal{P}(\mathcal{L})^2$, then we may find $\mathcal{T} \subseteq \mathcal{L}$ such that $\mathcal{T}$ is realized iff $\mathcal{I}$ is realized.

Proof. Proof is constructive. Let

$$\mathcal{I} = \{\langle \emptyset, \tau \rangle | \tau \in \mathcal{T}\}.$$

Result is immediate.

Next suppose we have $\mathcal{I}$. Let

$$\mathcal{T} = \{(\&\Gamma) \to (\bigvee \Theta) | \langle \Gamma, \Theta \rangle \in \mathcal{I}\}.$$

■

We have therefore established that realization of theories and realization of inferential relations coincide. But this is a rather weak notion of realization. After all, if we simply let $\vdash_0 = \mathcal{P}(\mathcal{L}_0)^2$ then any theory or inferential relation will be realized vacuously. Next I introduce the notion of a "proper" theory and base consequence relation. I shall prove that this can be realized exactly.

Definition A.1.13 (Proper Theory)**.** We say that a theory is **proper** if it is closed according to the following three rules and four substitution properties.[4]

R1: &-Composition If $\ulcorner A \urcorner, \ulcorner B \urcorner \in \mathcal{T}$ then $\ulcorner A \& B \urcorner \in \mathcal{T}$.

R2: &-Decomposition If $\ulcorner A \& B \urcorner \in \mathcal{T}$ then $\ulcorner A \urcorner, \ulcorner B \urcorner \in \mathcal{T}$.

R3: Distribution $\ulcorner A \vee (B \& C) \urcorner \in \mathcal{T}$ iff $\ulcorner (A \vee B) \& (A \vee C) \urcorner \in \mathcal{T}$.

S1: Conditional Let σ be an instance of a sub-formula of τ. Then $\tau \in \mathcal{T}$ for $\sigma = \ulcorner A \to B \urcorner$ iff $\tau[\sigma/\sigma'] \in \mathcal{T}$ for $\sigma' = \ulcorner \neg A \vee B \urcorner$.

S2: De Morgan's-1 Let σ be an instance of a sub-formula of τ. Then $\tau \in \mathcal{T}$ for $\sigma = \ulcorner \neg (A \vee B) \urcorner$ iff $\tau[\sigma/\sigma'] \in \mathcal{T}$ for $\sigma' = \ulcorner \neg A \& \neg B \urcorner$.

S3: De Morgan's-2 Let σ be an instance of a sub-formula of τ. Then $\tau \in \mathcal{T}$ for $\sigma = \ulcorner \neg (A \& B) \urcorner$ iff $\tau[\sigma/\sigma'] \in \mathcal{T}$ for $\sigma' = \ulcorner \neg A \vee \neg B \urcorner$.

S4: Involution Let σ be an instance of a sub-formula of τ. Then $\tau \in \mathcal{T}$ for $\sigma = \ulcorner \neg\neg B \urcorner$ iff $\tau[\sigma/\sigma'] \in \mathcal{T}$ for $\sigma' = \ulcorner B \urcorner$.

[4] I don't introduce explicitly—though a stricter treatment would require me to—that, for example, theories are required to obey commutivity for '&' and '∨'.

Note that normally when "theories" in the above sense (in sentential or first-order logic) are introduced they are usually introduced not only as a set of sentences, but a set of sentences which is closed in some sense. Usually they are thought to be closed under, for example, classical consequence. All of the above properties are entailed by classical consequence, but note that the criteria for proper theory I have given is strictly *weaker* than classical logic. That is, not all proper theories are theories of classical logic. A symptom of this discrepancy is that proper theories are closed under neither *disjunctive syllogism* nor *modus ponens*.[5]

Example A.1.14. Suppose $\ulcorner A \rightarrow B \urcorner, \ulcorner A \urcorner \in \mathcal{T}$. There is no guarantee that $\ulcorner B \urcorner \in \mathcal{T}$. The only rule that could yield B is R2, but there is no rule or substitution property that yields a sentence with B as a conjunct (the only possibilities are R1 (which would require an independent way to get B) or S2).

Next I show that "proper" theories have an interesting property: they contain the conjunctive normal form of all of their sentences. First a lemma.

Lemma A.1.15. *Suppose $\mathcal{T}$ is proper. Then if $\tau \in \mathcal{T}$ an equivalent sentence σ is also in $\mathcal{T}$ such that σ is either a conjunction or a disjunction.*[6]

Proof. Suppose $\tau \in \mathcal{T}$. If τ is a disjunction or conjunction we are done. It remains that τ may be a conditional or negated sentence. I proceed by induction on complexity. Suppose our result holds for sentences of complexity strictly less than n. Now suppose τ is of complexity n. If τ is a conditional then it is of the form $A \rightarrow B$. By S1 $\ulcorner A \rightarrow B \urcorner \in \mathcal{T}$ iff $\ulcorner \neg A \vee B \urcorner \in \mathcal{T}$. If τ is a negated sentence is of the form $\neg A$. By hypothis if $\ulcorner A \urcorner \in \mathcal{T}$ then there is an equivalent disjunction or conjunction in $\mathcal{T}$. If it is a disjunction of the form $B \vee C$ then by S3 $\ulcorner \neg B \& \neg C \urcorner \in \mathcal{T}$. If it is a conjunction of the form $B \& C$ then by S2 $\ulcorner \neg B \vee \neg C \urcorner \in \mathcal{T}$. ■

Lemma A.1.16. *Suppose $\mathcal{T}$ is proper and $\tau \in \mathcal{T}$. Then we may find an equivalent sentence $\sigma \in \mathcal{T}$ such that σ is either a disjunction of literals or has '&' as its main connective.*

Proof. I proceed by induction on the complexity of τ. By our previous lemma we need only consider cases in which τ is a conjunction or disjunction. If τ is atomic we are done, so

[5]Note that given S1 and S4, disjunctive syllogism and modus ponens are equivalent.

[6]Note that here "equivalent" means that proper theories contain both or none of these sentences. That is, σ and τ are equivalent iff for an arbitrary theory $\mathcal{T}$: $\sigma \in \mathcal{T}$ iff $\tau \in \mathcal{T}$.

assume the result holds for τ of logical complexity less than n and now let τ be of complexity n. If τ is a conjunction we are done again, so τ let τ be a disjunction of the form $\tau = \ulcorner A \vee B \urcorner$. By hypothesis there is a ρ and ρ' equivalent to A and B, respectively which are either the disjunction of literals or have '&' as their main connective. Suppose ρ and ρ' are both disjunctions of literals, then $\sigma = \ulcorner \rho \vee \rho' \urcorner \in \mathcal{T}$ (clearly the disjunction of disjunctions of literals is also a disjunction of literals).

Next, suppose instead without loss of generality that ρ has '&' as its main connective, i.e. $\rho = \ulcorner C \& D \urcorner$. Then $\ulcorner A \vee (C \& D) \urcorner \in \mathcal{T}$. By R3 we are guaranteed $\ulcorner (A \vee C) \& (A \vee D) \urcorner \in \mathcal{T}$. So let $\sigma = \ulcorner (A \vee C) \& (A \vee D) \urcorner$. ■

Theorem A.1.17. *Let $\mathcal{T}$ be a proper theory. Then*

$$\tau \in \mathcal{T} \Leftrightarrow CNF(\tau) \in \mathcal{T}.$$

Proof. Let $\mathcal{T}$ a proper theory. We show $\tau \in \mathcal{T} \Leftrightarrow CNF(\tau) \in \mathcal{T}$ by induction on logical complexity of τ.

In the base case τ is an atom and so $\tau = CNF(\tau)$. Therefore suppose that our result holds for τ of logical complexity strictly less than n and now suppose that τ is of complexity n. By our previous lemma we need only consider disjunctions of literals and sentences that have '&' as their main connective.

Suppose τ is a conjunction of the form $A \& B$. By R2 we have $\ulcorner A \urcorner, \ulcorner B \urcorner \in \mathcal{T}$ and thus by our inductive hypothesis $CNF(A), CNF(B) \in \mathcal{T}$. By R1 we have $\ulcorner CNF(A) \& CNF(B) \urcorner \in \mathcal{T}$. Clearly $\ulcorner CNF(A) \& CNF(B) \urcorner$ is in conjunctive normal form.

It remains then that τ is a disjunction of literals. In this case τ is already in CNF. ■

What this previous theorem in fact provides is a better characterization of proper theories. We might better understand proper theories as theories which obey R1, R2, and which contains a sentence if and only if it contains the CNF of that sentence.

Corollary A.1.18. *$\mathcal{T}$ is proper iff $\mathcal{T}$ satisfies R1, R2, and the following condition:*

$$\tau \in \mathcal{T} \Leftrightarrow CNF(\tau) \in \mathcal{T}.$$

Definition A.1.19 (Representative)**.** Suppose that $\vdash$ realizes inferential relation $\mathcal{I}$ exactly. Then we may call $\vdash_0$ (i.e. the base consequence relation of $\vdash$) the **representative** of inferential relation $\mathcal{I}$.

Suppose that $\vdash$ realizes $\mathcal{T}$ exactly. Let $\mathcal{I}$ be an equivalent inferential relation and thus $\vdash_0$ its representative. We call the following the **representative** of $\mathcal{T}$:

$$\{\neg p_1 \vee \cdots \vee \neg p_n \vee q_1 \vee \cdots \vee q_m \in \mathcal{L} | p_1, \ldots, p_n \vdash_0 q_1, \ldots, q_m\}.$$

I next demonstrate exactly how these "representatives" are constructed. For each $\langle \Gamma, \Theta \rangle \in \mathcal{I}$ we may determine exactly which atomic implications are required to realize $\Gamma \vdash \Theta$. Via Corollary A.1.8 we have $\Gamma \vdash \Theta$ iff $DNF(A) \vdash CNF(A)$, i.e.

$$\&(\Gamma_1 \cup \neg\Theta_1) \vee \cdots \vee \&(\Gamma_n \cup \neg\Theta_n) \vdash \bigvee(\neg\Delta_1 \cup \Lambda_1) \& \cdots \& \bigvee(\neg\Delta_m \cup \Lambda_m).$$

It follows that (for $1 \leq i \leq n$ and $1 \leq j \leq m$) that:

$$\Gamma_i, \Delta_j \vdash_0 \Theta_i, \Lambda_j.$$

If we take the union of all such sets of atomic implications (for each member of $\mathcal{I}$) then we have our representative.

Since proper $\mathcal{T}$ are characterized by the CNF of each of its members, we may construct its representative in the same manner.

Theorem A.1.20 (Representation Theorem 1)**.** *Let $\mathcal{T}$ (or $\mathcal{I}$) be arbitrary. Then we may specify $\vdash_0$ such that $\vdash$ realizes $\mathcal{T}$ (or $\mathcal{I}$).*

I actually show something stronger. Namely, I show how to expand $\mathcal{T}$ (and likewise $\mathcal{I}$) into minimal, proper theories. I also, thereby, show how to find the smallest $\vdash_0$ that realizes a theory (when it fails to realize that theory exactly). However, in order to minimize the amount of work done here, I prove the second representation theorem first, from which the first follows as a special case.

Theorem A.1.21 (Representation Theorem 2)**.** *Let $\mathcal{T}$ be proper (or $\mathcal{I}$ be equivalent to a proper $\mathcal{T}$). Then we may specify $\vdash_0$ such that $\vdash$ realizes $\mathcal{T}$ (or $\mathcal{I}$)* exactly*.*

Put otherwise: $\mathcal{T}$ may be realized exactly iff *$\mathcal{T}$ is proper.*

Proof. ($\Leftarrow$) Suppose $\mathcal{T}$ is proper. We must find $\vdash_0$ such that $\tau \in \mathcal{T}$ iff $\vdash \tau$.

Let us construct $\vdash_0$ according to the above procedure outline for generating a representative for $\mathcal{T}$.

Let $\vdash_0$ be specified according to the above procedure. For each member of σ of $\mathcal{T}$ with

$$CNF(\sigma) = \bigvee(\neg\Delta_1 \cup \Lambda_1)\& \cdots \& \bigvee(\neg\Delta_m \cup \Lambda_m),$$

we stipulate that $\vdash_0$ contain

$$\Delta_i \vdash \Lambda_j,$$

for $1 \leq i \leq m$ and $1 \leq j \leq m$. Next, given that $\tau \in \mathcal{T}$ iff $CNF(\tau) \in \mathcal{T}$ (via Theorem A.1.17), clearly $\vdash CNF(\tau)$ since we have stipulated that $\vdash_0$ contains the materials for constructing $\vdash CNF(\tau)$—and I note that this connection is biconditional. Further, I note that $\vdash CNF(\tau)$ iff $\vdash \tau$ (via Proposition A.1.7). It therefore follows that

$$\tau \in \mathcal{T} \Leftrightarrow \vdash \tau.$$

That is $\vdash$ realizes $\mathcal{T}$ *exactly.*

($\Rightarrow$) Let $\mathcal{T}$ be a theory and suppose $\vdash$ realizes $\mathcal{T}$ *exactly.* We wish to show that $\mathcal{T}$ is proper. Via corollary A.1.18 it is sufficient to show that $\mathcal{T}$ obeys R1, R2 and that

$$\tau \in \mathcal{T} \Leftrightarrow CNF(\tau) \in \mathcal{T}.$$

The last stipulation is immediate since we have shown (via Proposition A.1.7) that

$$\vdash \tau \Leftrightarrow \vdash CNF(\tau).$$

Next, I note that:

$$\vdash A\&B \Leftrightarrow \vdash A \text{ and } \vdash B,$$

It follows that

$$\ulcorner A\&B \urcorner \in \mathcal{T} \Leftrightarrow A, B \in \mathcal{T}.$$

That is, $\mathcal{T}$ satisfies R1 and R2. ■

Now, I return to a proof of the first representation theorem. I first, however, introduce the following definition, which streamlines proof of the result and is of independent interest.

Definition A.1.22 (Minimal, Proper Theory)**.** Let $\mathcal{T}$ be an arbitrary theory. Let $CNF(\mathcal{T})$ specify the set that contains all and only the CNF sentences of every sentence in $\mathcal{T}$:

$$CNF(\mathcal{T}) = \{CNF(\tau)|\tau \in \mathcal{T}\}.$$

We call $\mathcal{S}$ the **Minimal, Proper Theory** of $\mathcal{T}$ iff $\mathcal{S}$ is proper and

$$CNF(A) \in \mathcal{S} \Leftrightarrow CNF(A) \in CNF(\mathcal{T}).$$

We can use a similar procedure to define the **Minimal, Proper Consequence Relation** of a consequence relation $\mathcal{I}$.

Now, here is the proof of the first Representation Theorem (A.1.20).

Proof. Let $\mathcal{T}$ be an arbitrary theory. Let $\tau \in \mathcal{T}$ be arbitrary. We wish to find $\vdash_0$ such that $\vdash \tau$.

Next let $\mathcal{S}$ be the *minimal, proper theory* of $\mathcal{T}$. It follows that $CNF(\tau) \in \mathcal{S}$. Now, from the proof of A.1.21 we are guaranteed the existence of $\vdash_0$ and $\vdash$ that realize $\mathcal{S}$ exactly. That is: $\vdash CNF(\tau)$. Further (via Theorem A.1.17) we have that $\tau \in \mathcal{S}$.[7] Thus $\vdash \tau$. ■

Corollary A.1.23. *Let $\mathcal{T}$ be an arbitrary theory, let $\mathcal{S}$ be its minimal, proper theory, and let $\vdash_0$ be such that $\vdash$ realizes $\mathcal{S}$* exactly*. It follows that $\vdash_0$ (and $\vdash$) is minimal with respect to consequence relations that satisfy $\mathcal{T}$.*

[7] Alternatively, via Proposition A.1.7, we have the next step directly.

A.1.3.1 Applications

I delay a more thorough discussion of the application of the representation theorem for later, but I would like to foreshadow now its use. What the representation theorems tell us is that a specification of atomic behavior is not only sufficient (but necessary) for specifying the behavior of logically complex implications. This means that given a theory, we may find the behavior of any of the sentences that appears within it and specify that sentence's role in terms of $\vdash_0$. If we think theories capable of implicitly defining the sentences that occur within them, then this is a powerful result for connecting the theory to a notion of meaning (i.e. role in implication).

Further, what the notion of representative (Definition A.1.19) gives us, is a way of specifying for proper theories, the material element of a consequence relation, that is, the relation between atomic sentences that must obtain for a theory (or consequence relation) to obtain.

A.1.4 Theories

I next explain ways in which we may limit what counts as a proper $\vdash_0$. At the start I introduced several constraints we may wish to place on $\vdash_0$, e.g. contractivity, reflexivity, monotonicity, flatness, etc. I explain some interesting features that result from these constraints as well explain how to implement so-far undiscussed constraints. In particular I aim to the prove the following results:[8]

1. $\vdash_0$ is monotonic iff $\vdash$ is monotonic.
2. $\vdash_0$ is contractive iff $\vdash$ is contractive.
3. $\vdash_0$ is flat iff $\vdash$ is flat.
4. Minimal, flat $\vdash_0$ is equivalent to classical logic (i.e. $\vdash_{LK}$).
5. Existence of a condition on $\vdash_0$ for which $\vdash$ is transitive (i.e. admits cut-elimination).
6. Theories: application of representation theorem

I prove these results in order. First a lemma.

[8](1),(3), and (4) are already well-established, though I go through the proofs here for the sake of completeness. Most of the machinery to establish (2) and (5) exists independently of my apparatus, though I am unaware of a discussion of exactly these results.

Lemma A.1.24. *We may weaken a sequent with arbitrary atoms iff we may weaken it with arbitrary sentences (of potential logical complexity):*

$$\forall \Delta_0, \Lambda_0 \subseteq \mathcal{L}_0(\Delta_0, \Gamma \vdash \Theta, \Lambda_0) \Leftrightarrow \forall \Delta, \Lambda \subseteq \mathcal{L}(\Delta, \Gamma \vdash \Theta, \Lambda).$$

Proof. The ($\Leftarrow$)-direction is immediate. The ($\Rightarrow$)-direction is also proven fairly quickly. Let Δ and Λ be arbitrary. We have that:[9]

$$\Delta, \Gamma \vdash \Theta, \Lambda \Leftrightarrow \&\Delta, \Gamma \vdash \Theta, \bigvee \Lambda.$$

We therefore have

$$DNF(\&\Delta), \Gamma \vdash \Theta, CNF(\bigvee \Lambda).$$

But this is just equivalent to

$$\&(\Gamma_1 \cup \neg\Theta_1) \vee \cdots \vee \&(\Gamma_n \cup \neg\Theta_n), \Gamma \vdash \Theta, \bigvee(\neg\Delta_1 \cup \Lambda_1) \& \cdots \& \bigvee(\neg\Delta_m \cup \Lambda_m).$$

It follows that (for $1 \leq i \leq n$ and $1 \leq j \leq m$) that:

$$\Gamma_i, \Delta_j, \Gamma \vdash_0 \Theta, \Theta_i, \Lambda_j.$$

Since $\Gamma_i, \Delta_j, \Theta_i, \Lambda_j$ are atomic (for $1 \leq i \leq n$ and $1 \leq j \leq m$), we have our result. ■

Theorem A.1.25. $\vdash_0$ *is monotonic iff* $\vdash$ *is monotonic.*

Proof. The ($\Leftarrow$) direction is immediate. The ($\Rightarrow$) direction follows via induction on proof height.

Let Δ, Λ be arbitrary. We must show $\Delta, \Gamma \vdash \Theta, \Lambda$ (given that $\vdash_0$ is atomic). Now, in the base case $\Gamma \vdash \Theta$ is atomic. Via Lemma A.1.24, we have our result.

Now suppose our result holds for proof trees of height strictly less than n and that $\Gamma \vdash \Theta$ is obtained via a proof tree of height n. It may come via any of our rules. I divide into two cases where $\Gamma \vdash \Theta$ comes either via a rule with one-top sequent or via a rule with two-top sequents.

Case 1: Suppose $\Gamma \vdash \Theta$ comes via a rule with one-top sequent:

[9] An independent proof of result—that doesn't take advantage of my previously introduced machinery—can be constructed via induction on logical complexity.

$$\frac{\Gamma^* \vdash \Theta^*}{\Gamma \vdash \Theta} *$$

By hypothesis we have $\Delta, \Gamma^* \vdash \Theta^*, \Lambda$. Via an application of the same rule (*) we have our result: $\Delta, \Gamma \vdash \Theta, \Lambda$.

Case 2: Suppose $\Gamma \vdash \Theta$ comes via a rule with two-top sequents:

$$\frac{\Gamma^* \vdash \Theta^* \qquad \Gamma^{**} \vdash \Theta^{**}}{\Gamma \vdash \Theta} **$$

By hypothesis we have $\Delta, \Gamma^* \vdash \Theta^*, \Lambda$ and $\Delta, \Gamma^{**} \vdash \Theta^{**}, \Lambda$. Via an application of the same rule (**) we have our result: $\Delta, \Gamma \vdash \Theta, \Lambda$.

$\blacksquare$

Next, I show that a similar result holds for contractivity.

Lemma A.1.26. *Suppose $\vdash$ obeys contraction of atoms. I.e. $p, p, \Gamma \vdash \Theta$ only if $p, \Gamma \vdash \Theta$ and $\Gamma \vdash \Theta, p, p$ only if $\Gamma \vdash \Theta, p$. It follows that $\vdash$ obeys contraction of arbitrary sentences, i.e. $A, A, \Gamma \vdash \Theta$ only if $A, \Gamma \vdash \Theta$ and $\Gamma \vdash \Theta, A, A$ only if $\Gamma \vdash \Theta, A$.*

Proof. I proceed via induction on the complexity of A. In the base case A is atomic and we have our result immediately. Therefore suppose our result holds when A is of logical complexity strictly less than n and now suppose A is of complexity n. A may be a conditional, conjunction, disjunction, or negated sentence.

If A is a conditional then it is of the form $B \to C$. Then if we have $B \to C, B \to C, \Gamma \vdash \Theta$, we must have:

$$\Gamma \vdash \Theta, B, B \tag{1}$$

$$C, \Gamma \vdash \Theta, B$$

$$C, C, \Gamma \vdash \Theta \tag{2}$$

Applying our inductive hypothesis to (1) and (2) yields

$$\Gamma \vdash \Theta, B$$

$$C, \Gamma \vdash \Theta$$

Via L$\to$ we have $B \to C, \Gamma \vdash \Theta$.

Likewise if we have $\Gamma \vdash \Theta, B \to C, B \to C$ then this must come from $B, B, \Gamma \vdash \Theta, C, C$. Via our inductive hypothesis we have $B, \Gamma \vdash \Theta, C$ and via R$\to$ we have $\Gamma \vdash \Theta, B \to C$.

The cases where A is either a conjunction or disjunction are handled analogously.

If A is a negation, i.e. of the form $\neg B$, then we may reason as follows. If we have $\neg B, \neg B, \Gamma \vdash \Theta$ then this must come via $\Gamma \vdash \Theta, B, B$. By hypothesis $\Gamma \vdash \Theta, B$ and via L$\neg$ we have $\neg B, \Gamma \vdash \Theta$. If we have $\Gamma \vdash \Theta, \neg B, \neg B$ then $\Gamma \vdash \Theta, \neg B$ follows in a similar fashion. ■

Theorem A.1.27. $\vdash_0$ *is contractive iff* $\vdash$ *is contractive.*

Proof. The ($\Leftarrow$) direction is immediate so I show the ($\Rightarrow$) direction. It is sufficient to show that contraction of atoms is preserved (i.e. that $\vdash$ allows contraction of atomic sentences). Lemma A.1.26 secures the result after that.

I proceed therefore via induction on proof height. In the base case $\Gamma \vdash \Theta$ is atomic, so we are done. Suppose, therefore, that $\Gamma \vdash \Theta$ admits contraction of atoms for proof trees of height strictly less than n and that $p, p, \Gamma \vdash \Theta$ and $\Gamma \vdash \Theta, p, p$ each come via proof trees of height n. They may come via any of our rules. Because p is atomic we can be certain that it is not the principal formula in our rule application. I treat two cases (where they come via a rule with one top sequent and where they come via a rule with two top sequents).

Case 1: Suppose $p, p, \Gamma \vdash \Theta$ (and $\Gamma \vdash \Theta, p, p$) come via a rule with one-top sequent:

$$\frac{p, p, \Gamma^* \vdash \Theta^* \text{ (or } \Gamma^* \vdash \Theta^*, p, p)}{\Gamma \vdash \Theta \text{ (or } \Gamma \vdash \Theta, p, p)} *$$

By hypothesis we have $p, \Gamma^* \vdash \Theta^*$ (and $\Gamma^* \vdash \Theta^*, p$). Via an application of the same rule (*) we have our result: $p, \Gamma \vdash \Theta$ (and $\Gamma \vdash \Theta, p$).

Case 2: Suppose $p, p, \Gamma \vdash \Theta$ (and $\Gamma \vdash \Theta, p, p$) come via a rule with two-top sequents:

$$\frac{p, p, \Gamma^* \vdash \Theta^* \text{ (or } \Gamma^* \vdash \Theta^*, p, p) \qquad p, p, \Gamma^{**} \vdash \Theta^{**} \text{ (or } \Gamma^{**} \vdash \Theta^{**}, p, p)}{p, p, \Gamma \vdash \Theta \text{ (or } \Gamma \vdash \Theta, p, p)} **$$

By hypothesis we have $p, \Gamma^* \vdash \Theta^*$ (and $\Gamma^* \vdash \Theta^*, p$) as well as $p, \Gamma^{**} \vdash \Theta^{**}$ (and $\Gamma^{**} \vdash \Theta^{**}, p$). Via an application of the same rule (**) we have our result: $p, \Gamma \vdash \Theta$ (and $\Gamma \vdash \Theta, p$).

■

Next, I show that $\vdash_0$ is flat iff $\vdash$ is flat, i.e.

$$\forall \Gamma_0, \Theta_0, p(\Gamma_0, p \vdash_0 p, \Theta_0) \Leftrightarrow \forall \Gamma, \Theta, A(\Gamma, A \vdash A, \Theta).$$

I have already shown that monotonicity is preserved so it remains to show that reflexivity in preserved in such contexts.

Lemma A.1.28. *If $\vdash_0$ is flat, then $\vdash$ is reflexive, i.e. $A \vdash A$ for all A.*

Proof. I proceed via induction on the complexity of A. If A is atomic we are done, so suppose the result holds when A is of complexity strictly less than n and now suppose A is of complexity n. It may be a conditional, conjunction, disjunction, or negated sentence.

Suppose A is a conditional of the form $B \to C$. By hypothesis we have $B \vdash C, B$ and $B, C \vdash C$. We derive:

$$\cfrac{\cfrac{C, B \vdash C \qquad B \vdash C, B}{B \to C, B \vdash C}\ \text{L}\!\to}{B \to C \vdash B \to C}\ \text{R}\!\to$$

$\vee$ and & are handled analogously. Therefore suppose that A is an negated sentence of the form $\neg B$. By hypothesis we have $B \vdash B$ and so

$$\cfrac{\cfrac{B \vdash B}{\neg B, B \vdash}\ \text{L}\neg}{\neg B \vdash \neg B}\ \text{R}\neg$$

■

The result is immediate.

Theorem A.1.29. *$\vdash_0$ is flat iff $\vdash$ is flat.*

Proof. Since flat $\vdash_0$ gives us reflexive $\vdash$ and such sequents will be monotonic we are done. ■

We have so far constructed nearly all of the machinery needed to demonstrate that minimal, flat $\vdash_0$ is equivalent to classical logic. In fact the result should be obvious given the following facts:

- Minimal, flat $\vdash_0$ obeys monotonicity, reflexivity, and *contraction* (i.e. these structural rules are admissible).
- Given monotonicity and contraction, the rules of my sequent calculus are equivalent to e.g. Gentzen's LK rules ($\vdash_{LK}$).

- In $\vdash_{LK}$ the sole axiom is idempotence (i.e. reflexivity). Since we are guaranteed $A \vdash A$ for all A it should be clear that we can reconstruct all such axioms.

Since all of these are well-attested (and in general the equivalence to LK is well-known) I will not spend much time proving this. Instead I'll briefly sketch for the reader unfamiliar with these results how the rules of of my sequent calculus and LK are equivalent given minimal, flat $\vdash_0$. First, I list here all of Gentzen's rules (I replace some of his symbols to keep these two systems distinct).

$$\frac{\Delta \vdash_{LK} \Lambda, A \qquad B, \Gamma \vdash_{LK} \Theta}{A \supset B, \Gamma \vdash_{LK} \Theta}\,\text{L}\supset \qquad \frac{A, \Gamma \vdash_{LK} \Theta, B}{\Gamma \vdash_{LK} A \supset B, \Theta}\,\text{R}\supset$$

$$\frac{\Gamma, A_i \vdash_{LK} \Theta}{\Gamma, A_1 \wedge A_2 \vdash_{LK} \Theta}\,\text{L}\wedge\ (i = 1 \text{ or } 2) \qquad \frac{\Delta \vdash_{LK} \Lambda, A \qquad \Gamma \vdash_{LK} \Theta, B}{\Gamma, \Delta \vdash_{LK} \Theta, \Lambda, A \wedge B}\,\text{R}\wedge$$

$$\frac{A, \Delta \vdash_{LK} \Lambda \qquad B, \Gamma \vdash_{LK} \Theta}{A \vee_{LK} B, \Gamma, \Delta \vdash_{LK} \Theta, \Lambda}\,\text{L}\vee_{LK} \qquad \frac{\Gamma \vdash_{LK} \Theta, A_i}{\Gamma \vdash_{LK} \Theta, A_1 \vee_{LK} A_2}\,\text{R}\vee_{LK}\ (i = 1 \text{ or } 2)$$

$$\frac{\Gamma \vdash_{LK} \Theta, A}{\sim A, \Gamma \vdash_{LK} \Theta}\,\text{L}\sim \qquad \frac{A, \Gamma \vdash_{LK} \Theta}{\Gamma \vdash_{LK} \Theta, \sim A}\,\text{R}\sim$$

Gentzen also includes the structural rules (I don't list permutation):

$$\frac{\Gamma \vdash_{LK} \Theta}{\Delta, \Gamma \vdash_{LK} \Theta, \Lambda}\,\text{MO} \qquad \frac{A, A, \Gamma \vdash_{LK} \Theta}{A, \Gamma \vdash_{LK} \Theta}\,\text{L-Contr.} \qquad \frac{\Gamma \vdash_{LK} \Theta, A, A}{\Gamma \vdash_{LK} \Theta, A}\,\text{R-Contr.}$$

And the axiom:

$$\frac{}{A \vdash_{LK} A}\,\text{Idem.}$$

It should be obvious how all proof trees of LK can be replicated. We can construct any axioms of LK given a minimal, flat $\vdash_0$. Further, any application of a structural rule (MO or Contraction) are guaranteed from the results in this section. It remains that we can find equivalent rule applications. I show why this is the case for conjunction (the other cases are either analogous or immediate).

I show any valid application of L$\wedge$ may be reproduced in my sequent calculus and vice-versa. Suppose we have:

$$\frac{\Gamma, A \vdash_{LK} \Theta}{\Gamma, A \wedge B \vdash_{LK} \Theta}\,\text{L}\wedge$$

In my sequent calculus, if we have $\Gamma, A \vdash \Theta$ then we also have $\Gamma, A, B \vdash \Theta$. Via L& we have $\Gamma, A\&B \vdash \Theta$.

Conversely, if we have:

$$\frac{\Gamma, A, B \vdash \Theta}{\Gamma, A\&B, \vdash \Theta}\text{L\&}$$

We may find an equivalent derivation in LK:

$$\dfrac{\dfrac{\dfrac{\Gamma, A, B \vdash_{LK} \Theta}{\Gamma, A \wedge B, B \vdash_{LK} \Theta}\text{L}\wedge}{\Gamma, A \wedge B, A \wedge B \vdash_{LK} \Theta}\text{L}\wedge}{\Gamma, A \wedge B \vdash_{LK} \Theta}\text{L-Contr.}$$

We may show something similar for conjunction in the succedent. Suppose we have:

$$\frac{\Delta \vdash_{LK} \Lambda, A \qquad \Gamma \vdash_{LK} \Theta, B}{\Gamma, \Delta \vdash_{LK} \Theta, \Lambda, A \wedge B}\text{R}\wedge$$

Then if we have $\Delta \vdash \Lambda, A$ and $\Gamma \vdash \Theta, \Lambda, B$ we also have $\Delta, \Gamma \vdash \Lambda, A$ and $\Delta, \Gamma \vdash \Theta, \Lambda, B$ (since $\vdash_0$ is monotonic). Thus via R& we have: $\Delta, \Gamma \vdash \Theta, \Lambda, A\&B$.

Conversely, if we have:

$$\frac{\Gamma \vdash \Theta, A \qquad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A\&B}\text{R\&}$$

We may find an equivalent derivation in LK:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash_{LK} \Theta, A \qquad \Gamma \vdash_{LK} \Theta, B}{\Gamma, \Gamma \vdash_{LK} \Theta, \Theta, A \wedge B}\text{R}\wedge}{\vdots}\text{L-Contr.}}{\Gamma \vdash_{LK} \Theta, \Theta, A \wedge B}\text{L-Contr.}}{\vdots}\text{R-Contr.}}{\Gamma \vdash_{LK} \Theta, A \wedge B}\text{R-Contr.}$$

From these considerations the following result is immediate:

Theorem A.1.30. $\vdash_0$ *is flat iff* $\vdash$ *is* **supra-classical** *in the sense that any implication of classical logic is included in* $\vdash$*, i.e.* $\vdash_{LK} \subseteq \vdash$.

This only demonstrates a rather limited version of supra-classicality (that the logic include all classically valid implications). Often, when we wish to characterize a consequence relation as supra-classical, we have something stronger in mind: that it obeys classical principles of reasoning.[10] That is, that $\vdash$ be (at least) monotonic and transitive. It is possible,

[10]Parenthetically: it is not so clear that one can cleanly separate these two things. But I do not address this here.

for example, for $\vdash$ to be supraclassical (and so $\vdash_0$ to be flat) even though $A \vdash B$ is not monotonic (if it occurs outside the fragment of $\vdash$ generated from the flat portion of $\vdash_0$). Likewise if we also have $B \vdash C$ (supposing this also occurs outside this fragment) we might have no guarantee that $A \vdash C$. To many, to character this as a supra-classical would be misleading (some might even refuse to call such a $\vdash$ a consequence relation at all). It will therefore be useful to explain how we can limit $\vdash_0$ to produce a *transitive* $\vdash$, i.e. a $\vdash$ that obeys cut.

Cut is typically formulated as:

$$\frac{\Delta \vdash \Lambda, A \qquad A, \Gamma \vdash \Theta}{\Delta, \Gamma \vdash \Theta, \Lambda}\text{ Cut}$$

It is easy to see, however, that together with a flat $\vdash_0$ imposing such a condition will result in monotonicity. It is easy to see why. Let Δ, Λ be arbitrary. Suppose $A \vdash B$. Clearly $A, \Delta \vdash \Lambda, A$. Via Cut we have:

$$\frac{A, \Delta \vdash \Lambda, A \qquad A \vdash B}{\Delta, A \vdash B, \Lambda}\text{ Cut}$$

Therefore, in order to keep these ideas notions separate it will be useful to examine instead a more restricted version of Cut:

$$\frac{\Gamma \vdash \Theta, A \qquad A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta}\text{ Shared-Cut}$$

Given MO and Contraction these two rules are obviously equivalent (again), so if one insists on monotonic $\vdash_0$ this choice makes no large difference.

It remains then to show what $\vdash_0$ must look like for this structural feature to hold. From previous proofs it should be apparent that we will be done if we find a condition such that: if $A, \Gamma_0 \vdash \Theta_0$ and $\Gamma_0 \vdash \Theta_0, A$ for atomic Γ_0, Θ_0, then $\Gamma_0 \vdash \Theta_0$.

The semantics I introduce in the next section make it significantly easier to formulate this condition, nevertheless I give a sense of what it looks like now. What we want is that anytime the atomic sequents used to construct $A, \Gamma \vdash \Theta$ and $\Gamma \vdash \Theta, A$ appear, that $\Gamma \vdash_0 \Theta$. But how do we specify such behavior? We can think of it as follows. Let A be some sentence, then anytime $DNF(A), \Gamma \vdash \Theta$ and $\Gamma \vdash \Theta, CNF(A)$, we have $\Gamma \vdash_0 \Theta$.

Recall that $DNF(A), \Gamma \vdash \Theta$ is generated by sequents with the following shape:

$$\Delta_1, \Gamma \vdash_0 \Theta, \Lambda_1$$
$$\vdots$$
$$\Delta_n, \Gamma \vdash_0 \Theta, \Lambda_n.$$

We can think of these as ordered pairs which, when combined with $\Gamma \vdash_0 \Theta$, generate a valid sequent. Thus, let us represent $DNF(A)$ on the left with the following set:

$$\{\langle \Delta_1, \Lambda_1 \rangle, \ldots, \langle \Delta_n, \Lambda_n \rangle\}.$$

Then we may represent $CNF(A)$ on the right as:

$$\{\langle \Pi, \Xi \rangle | \delta_i \in \Delta_i, \lambda_i \in \Lambda_i \text{ and } \textit{exactly} \text{ one of either } \delta_i \in \Xi \text{ or } \lambda_i \in \Pi \text{ (for } 1 \leq i \leq n)\}.$$

I.e. take one element from exactly one of the sets in each $\langle \Delta_i, \Lambda_i \rangle$ to construct each $\langle \Pi, \Xi \rangle$. If we enumerate this set as $\langle \Pi_1, \Xi_1 \rangle, \ldots, \langle \Pi_m, \Xi_m \rangle$, then $\Gamma \vdash \Theta, CNF(A)$ will come via:

$$\Pi_1, \Gamma \vdash_0 \Theta, \Xi_1$$
$$\vdots$$
$$\Pi_m, \Gamma \vdash_0 \Theta, \Xi_m.$$

Definition A.1.31 (Transitivity)**.** Let $A \subseteq \mathcal{P}(\mathcal{L}_0)$ be

$$A = \{\langle \Delta_1, \Lambda_1 \rangle, \ldots, \langle \Delta_n, \Lambda_n \rangle\},$$

and let $S \subseteq \mathcal{P}(\mathcal{L}_0)$ be

$$S = \{\langle \Pi, \Xi \rangle | \langle \Delta_i, \Lambda_i \rangle \in A, \delta_i \in \Delta_i, \lambda_i \in \Lambda_i \text{ and } \textit{exactly} \text{ one of either } \delta_i \in \Xi \text{ or } \lambda_i \in \Pi \text{ (for } 1 \leq i \leq n)\}.$$

Let us enumerate S as $\langle \Pi_1, \Xi_1 \rangle, \ldots, \langle \Pi_m, \Xi_m \rangle$. Let Γ, Θ be arbitrary. If whenever

$$\Delta_1, \Gamma \vdash_0 \Theta, \Lambda_1$$
$$\vdots$$
$$\Delta_n, \Gamma \vdash_0 \Theta, \Lambda_n.$$

and

$$\Pi_1, \Gamma \vdash_0 \Theta, \Xi_1$$
$$\vdots$$
$$\Pi_m, \Gamma \vdash_0 \Theta, \Xi_m.$$

then

$$\Gamma \vdash_0 \Theta,$$

then we say that $\vdash_0$ is **transitive** or satisfies **transitivity**.

Lemma A.1.32. *Suppose $\vdash_0$ satisfies transitivity. Then whenever $\Gamma_0, \Theta_0 \subseteq \mathcal{L}_0$ are such that $A, \Gamma_0 \vdash \Theta_0$ and $\Gamma_0 \vdash \Theta_0, A$ we have $\Gamma_0 \vdash \Theta_0$.*

Proof. Immediate from definition. Let A be in DNF which has the form:

$$\&(\Delta_1 \cup \neg\Lambda_1) \vee \cdots \vee \&(\Delta_n \cup \neg\Lambda_n), \Gamma_0 \vdash \Theta_0.$$

Clearly this comes via

$$\Delta_1, \Gamma \vdash_0 \Theta, \Lambda_1$$
$$\vdots$$
$$\Delta_n, \Gamma \vdash_0 \Theta, \Lambda_n.$$

in $\vdash_0$. Next, by supposing we have $\Gamma_0 \vdash \Theta_0, A$, i.e.

$$\Gamma_0 \vdash \Theta_0, \&(\Delta_1 \cup \neg\Lambda_1) \vee \cdots \vee \&(\Delta_n \cup \neg\Lambda_n).$$

I show how we can use this to construct S as in Definition A.1.31.

Since our rules are reversible, we therefore have:

$$\Gamma_0 \vdash \Theta_0, \&(\Delta_1 \cup \neg\Lambda_1), \ldots, \&(\Delta_n \cup \neg\Lambda_n).$$

Let us enumerate each Δ_i as $\delta_{i,1}, \ldots, \delta_{i,k_i}$ and each Λ_i as $\lambda_{i,1}, \ldots, \lambda_{i,k_i}$. If we focus on the final disjunction, then decompositions yields:

$$\begin{array}{r}
\Gamma_0 \vdash \Theta_0, \&(\Delta_1 \cup \neg\Lambda_1), \ldots, \&(\Delta_{n-1} \cup \neg\Lambda_{n-1}), \delta_{n,1} \\
\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\
\Gamma_0 \vdash \Theta_0, \&(\Delta_1 \cup \neg\Lambda_1), \ldots, \&(\Delta_{n-1} \cup \neg\Lambda_{n-1}), \delta_{n,k_n} \\
\lambda_{n,1}, \Gamma_0 \vdash \Theta_0, \&(\Delta_1 \cup \neg\Lambda_1), \ldots, \&(\Delta_{n-1} \cup \neg\Lambda_{n-1}) \qquad \\
\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\
\lambda_{n,k_n}, \Gamma_0 \vdash \Theta_0, \&(\Delta_1 \cup \neg\Lambda_1), \ldots, \&(\Delta_{n-1} \cup \neg\Lambda_{n-1}). \qquad
\end{array}$$

Repeating this procedure for each of the above generated sequents, it is clear that we can construct a set

$S = \{\langle \Pi, \Xi \rangle | \langle \Delta_i, \Lambda_i \rangle \in A, \delta_i \in \Delta_i, \lambda_i \in \Lambda_i$ and *exactly* one of either $\delta_i \in \Xi$ or $\lambda_i \in \Pi$ (for $1 \leq i \leq n$)$\}$.

We are therefore done. ■

Theorem A.1.33. $\vdash_0$ *is transitive iff Shared-Cut is admissible:*

$$\frac{\Gamma \vdash \Theta, A \qquad A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta}\ \textit{Shared-Cut}$$

Proof. ($\Rightarrow$) Suppose $\vdash_0$ is transitive. We show that cut is admissible by induction on proof height. We suppose that proofs assume a normal form under which the principal formula that is to be cut is constructed first and other rule applications are applied afterwards. In the base case $p, \Gamma \vdash \Theta$ and $\Gamma \vdash \Theta, p$ are atomic so we are done. Therefore suppose that the result holds for proof trees of height strictly less than n. Now suppose that $A, \Gamma \vdash \Theta$ and $\Gamma \vdash \Theta, A$ come from proof trees of at most height n. Regardless of which rule they issue from, we may use our inductive hypothesis to "cut" A and then via an application of the same rule obtain $\Gamma \vdash \Theta$.

($\Leftarrow$) Immediate from previous lemma. ■

What we have therefore are the tools for restricting $\vdash_0$ as we see fit. We may either restrict it by imposing certain structural conditions on $\vdash_0$ (monotonicity, contraction, flatness, transitivity, etc.) from which we know certain structural features of $\vdash$ to follow. Or we may restrict $\vdash_0$ so that certain theories or inferential relations are realized in $\vdash$. Together these tools will give us powerful means for exploring how, to what degree, and in what sense new sentential variables may be defined in a logic.

A.2 Semantics

In this section I introduce the inferential role semantics and prove that the proof theory (i.e. sequent calculus) of the previous section is *sound* and *complete* with respect to this semantics.

A.2.1 Inferential Spaces

Definition A.2.1 (Inferential Space)**.** Given a language $\mathcal{L}$ (of potential logical complexity), an **inferential space** is understood to be the set of ordered pairs of multi-sets of $\mathcal{L}$:[11]

$$\mathbf{P} = \mathcal{P}(\mathcal{L})^2.$$

We call each "point" (of the form $\langle X, Y \rangle$, where $X, Y \subseteq \mathcal{L}$) an **implication**.

Each inferential space $\mathbf{P}$ comes with a privileged subset of implications: the **good implications**.

$$\mathbb{I} \subseteq \mathbf{P}.$$

NB: I have so far introduced *no constraints* on what $\mathbb{I}$ must look like (nor on what $\mathbf{P}$) must look like. I have left it entirely open how each of these spaces is constituted.[12]

[11]See n.1 for more information on my choice to use multi-sets.

[12]Compare my treatment of $\vdash_0$ in the Appendix A.

Definition A.2.2 (fuission)**.** There is a single operation on **P** known as **fuission**, '$\sqcup$'. If $A = \langle \Gamma, \Theta \rangle$ and $B = \langle \Delta, \Lambda \rangle$, then[13]

$$A \sqcup B =_{df.} \langle \Gamma \cup \Delta, \Theta \cup \Lambda \rangle.$$

I also generalize '$\sqcup$' as an operation over subsets of **P**. If $X, Y \subseteq \mathbf{P}$, then:

$$X \sqcup Y = \{x \cup y | x \in X, y \in Y\}.$$

Definition A.2.3 (vee)**.** I next define a function which will eventually help to specify the inferential role of sentences. '$\curlyvee$' (pronounced "vee") is a function from subsets of **P** to subsets of **P**. '$\curlyvee$' tells us the role an ordered pair makes to good implication. Thus if $X = \langle \Gamma, \Delta \rangle$, then:

$$X^{\curlyvee} = \{\langle \Gamma, \Delta \rangle\}^{\curlyvee} =_{df.} \{\langle \Delta, \Lambda \rangle | \langle \Gamma, \Delta \rangle \sqcup \langle \Delta, \Lambda \rangle \in \mathbb{I}\}.$$

When X is a singleton, we may drop the set brackets where lack of ambiguity allows. In the case where $X \subseteq \mathbf{P}$ and is not a singleton, then '$\curlyvee$' is defined as:

$$X^{\curlyvee} =_{df.} \{\langle \Delta, \Lambda \rangle | \forall \langle \Gamma, \Theta \rangle \in X(\langle \Gamma, \Theta \rangle \sqcup \langle \Delta, \Lambda \rangle \in \mathbb{I})\}.$$

Definition A.2.4 (Closure)**.** '$\curlyvee$' allows us to define a closure operation. A set of implications $X \subseteq \mathbf{P}$ is said to be **closed** iff

$$X^{\curlyvee\curlyvee} = X.$$

I now prove that '$\curlyvee$' is in fact a *closure* operation in the way described. Traditionally *closure operations* $Cl(\cdot)$ are said to satisfy three properties:

Extensive: A closure contains that which it is the closure of: $X \subseteq Cl(X)$.
Idempotent: The closure of a closure is the same as the closure: $Cl(Cl(X)) = Cl(X)$.
Monotone: If $X \subseteq Y$, then $Cl(X) \subseteq Cl(Y)$.

Proposition A.2.5. '$\curlyvee\curlyvee$' is a closure operation.

[13]Note that '$\cup$' is here understood to be multi-set union, i.e. in this context $\{a\} \cup \{a\} = \{a, a\}$.

Proof. Let $X \subseteq \mathbf{P}$ be arbitrary. We must show that '$\curlyvee\curlyvee$' serves as a closure operation on such X.

Clearly '$\curlyvee\curlyvee$' is extensive. Let $\langle \Gamma, \Delta \rangle \in X$ be arbitrary. Now if $\langle x_1, x_2 \rangle \in X^{\curlyvee}$ then $\langle \Gamma, \Delta \rangle \sqcup \langle x_1, x_2 \rangle \in \mathbb{I}$. Since $\langle x_1, x_2 \rangle$ was chosen arbitrarily it follows that $\langle \Gamma, \Delta \rangle \in X^{\curlyvee\curlyvee}$. Thus:

$$X \subseteq X^{\curlyvee\curlyvee}.$$

'$\curlyvee\curlyvee$' is also idempotent. Since we have already shown it to be extensive, we know that $X^{\curlyvee\curlyvee} \subseteq X^{\curlyvee\curlyvee\curlyvee\curlyvee}$. Thus we need only show the converse, i.e. $X^{\curlyvee\curlyvee\curlyvee\curlyvee} \subseteq X^{\curlyvee\curlyvee}$. Let $a \in X^{\curlyvee}$ be arbitrary and $b \in X^{\curlyvee\curlyvee\curlyvee\curlyvee}$ be arbitrary. We must show $a \sqcup b \in \mathbb{I}$. We know that $a \in X^{\curlyvee\curlyvee\curlyvee}$. Thus, $a \sqcup b \in \mathbb{I}$ and so $a \in X^{\curlyvee\curlyvee}$. It follows that:

$$X^{\curlyvee\curlyvee\curlyvee\curlyvee} = X^{\curlyvee\curlyvee}.$$

Finally, '$\curlyvee\curlyvee$' is monotone. Let $X \subseteq Y \subseteq \mathbf{P}$. Now if $a \in Y^{\curlyvee}$ then for arbitrary $b \in Y$ we have that $a \sqcup b \in \mathbb{I}$. It therefore follows that $a \in X^{\curlyvee}$. Now let $c \in X^{\curlyvee\curlyvee}$ be arbitrary. Clearly $a \sqcup c \in \mathbb{I}$ and thus $c \in Y^{\curlyvee\curlyvee}$. It follows that:

$$X^{\curlyvee\curlyvee} \subseteq Y^{\curlyvee\curlyvee}.$$

■

Three corollaries follow immediately.

Corollary A.2.6. $X^{\curlyvee} = X^{\curlyvee\curlyvee\curlyvee}$.

Corollary A.2.7. *If* $X \subseteq Y$ *then* $Y^{\curlyvee} \subseteq X^{\curlyvee}$.

The final corollary is a result of the "extensive" property of closure operations.

Corollary A.2.8. $X \subseteq \mathbb{I}$ *iff* $X^{\curlyvee\curlyvee} \subseteq \mathbb{I}$.

The important or interesting feature of the previous corollary is that $X \in \mathbb{I}$ (i.e. X is a *good* implication) iff $X^{\curlyvee\curlyvee} \subseteq \mathbb{I}$.

Now I introduce the notion of a "proper inferential role".

Definition A.2.9 (Proper Inferential Role)**.** A **proper inferential role (PIR)** is a double $\langle X, Y\rangle$ such that X and Y are each *closed*—in the sense described above—subsets of $\mathbf{P}$ (i.e. $X^{\curlyvee\curlyvee} = X$ and $Y^{\curlyvee\curlyvee} = Y$). We should think of $\langle X, Y\rangle$ as specifying some inferential role and think of X as specifying that premissory role and Y as specifying that conclusory role.

As a convention if $A = \langle X, Y\rangle$ is an inferential role, then I write A^P to refer to X and A^C to refer to Y.

I call these inferential roles "proper" in order to distinguish it from a more informal notion of inferential role which need not be closed in the relevant sense.

Next, I introduce some important semantic notions.

A.2.2 Semantics

Definition A.2.10 (Models)**.** A **model** is a tuple $\langle \mathbf{P}, \mathbb{I}, [\![\cdot]\!]\rangle$ consisting of an language $\mathcal{L}$ and inferential space over that language $\mathbf{P}$, a privileged set of good implications $\mathbb{I}$, and an interpretation function $[\![\cdot]\!]$ (to be defined next) which interprets sentences in the language as inferential roles in the model.

.

Definition A.2.11 (Interpretation Function)**.** An interpretation function $[\![\cdot]\!]$ maps sentences in $\mathcal{L}$ to proper inferential roles in models (i.e. ordered pairs of closed sets of implications). If $A \in \mathcal{L}$ is atomic, then A is interpreted as follows:

$$[\![A]\!] =_{df.} \langle \langle \{A\}, \emptyset\rangle^{\curlyvee}, \langle \emptyset, \{A\}\rangle^{\curlyvee}\rangle.$$

My semantic definitions follow:

$$\begin{aligned}
[\![A \& B]\!] &=_{df.} \langle ((([\![A]\!]_P)^{\curlyvee} \sqcup ([\![B]\!]_P)^{\curlyvee})^{\curlyvee}, [\![A]\!]_C \cap [\![B]\!]_C\rangle, \\
[\![A \vee B]\!] &=_{df.} \langle [\![A]\!]_P \cap [\![B]\!]_P, (([\![A]\!]_C)^{\curlyvee} \sqcup ([\![B]\!]_C)^{\curlyvee})^{\curlyvee}\rangle, \\
[\![A \to B]\!] &=_{df.} \langle [\![A]\!]_C \cap [\![B]\!]_P, (([\![A]\!]_P)^{\curlyvee} \sqcup ([\![B]\!]_C)^{\curlyvee})^{\curlyvee}\rangle, \\
[\![\neg A]\!] &=_{df.} \langle [\![A]\!]_C, [\![A]\!]_P\rangle.
\end{aligned}$$

It is a legitimate question whether our semantic definitions successfully pick out *inferential roles* (in the sense made clear in Definition A.2.9). As I've already proven, for arbitrary $X \in \mathbf{P}$, we have that $X^{\curlyvee}$ is closed. This clarifies how many of our semantic definitions yield closed sets. But it is not immediately obvious why $X \cap Y$ will be closed even if both X and Y are. We must therefore provide a short proof of this. First, a useful lemma:

Lemma A.2.12. *Suppose X and Y are closed, then:*

$$X \cap Y = (X^{\curlyvee} \cup Y^{\curlyvee})^{\curlyvee}.$$

Proof. Let $a \in X \cap Y$. Let $b \in X^{\curlyvee} \cup Y^{\curlyvee}$. Either $b \in X^{\curlyvee}$ or $b \in Y^{\curlyvee}$. Without loss of generality, we have that $a \sqcup b \in \mathbf{P}$ (since $a \in X$ and $a \in Y$). It therefore follows that $a \in (X^{\curlyvee} \cup Y^{\curlyvee})^{\curlyvee}$.

Conversely, suppose $a \in (X^{\curlyvee} \cup Y^{\curlyvee})^{\curlyvee}$. Let $b \in (X^{\curlyvee} \cup Y^{\curlyvee})$. Either $b \in X^{\curlyvee}$ or $b \in Y^{\curlyvee}$. In either case $a \sqcup b \in \mathbf{P}$. Thus, $a \in X^{\curlyvee\curlyvee}$ and $a \in Y^{\curlyvee\curlyvee}$. Since X and Y are closed it follows that $a \in X$ and $a \in Y$, i.e. $a \in X \cap Y$. ■

Proposition A.2.13. If X and Y are closed, then so is $X \cap Y$.

Proof. We must show $(X \cap Y)^{\curlyvee\curlyvee} = X \cap Y$. From our lemma we have that:

$$(X \cap Y)^{\curlyvee\curlyvee} = (X^{\curlyvee} \cup Y^{\curlyvee})^{\curlyvee\curlyvee\curlyvee}.$$

We also know from a previous proof that

$$(X^{\curlyvee} \cup Y^{\curlyvee})^{\curlyvee\curlyvee\curlyvee} = (X^{\curlyvee} \cup Y^{\curlyvee})^{\curlyvee}.$$

Finally, applying the previous lemma again, we have:

$$(X^{\curlyvee} \cup Y^{\curlyvee})^{\curlyvee} = X \cap Y.$$

Thus,

$$(X \cap Y)^{\curlyvee\curlyvee} = X \cap Y.$$

. ■

We can also prove an interesting feature of our semantic definitions. This requires two lemmas, however, the second of which more or less guarantees the result:

Lemma A.2.14.

$$X^{\curlyvee\curlyvee} \sqcup Y^{\curlyvee\curlyvee} \subseteq (X \sqcup Y)^{\curlyvee\curlyvee}.$$

Proof. Let $a \in X^{\curlyvee\curlyvee}$ and $b \in Y^{\curlyvee\curlyvee}$. Next let $c \in (X \sqcup Y)^{\curlyvee}$. We must show that $a \sqcup b \sqcup c \in \mathbb{I}$.

Next let $x \in X$ and $y \in Y$. It follows that $c \sqcup x \sqcup y \in \mathbb{I}$. It thus follows that $c \sqcup y \in X^{\curlyvee}$. It therefore follows that $a \sqcup c \sqcup y \in \mathbb{I}$. Since $y \in Y$ it follows that $a \sqcup c \in Y^{\curlyvee}$ and thus $a \sqcup b \sqcup c \in \mathbb{I}$. Thus $a \sqcup b \in (X \sqcup Y)^{\curlyvee\curlyvee}$. ■

Before preceding I note that a significant corollary follows from the previous lemma.

Corollary A.2.15. $(X^{\curlyvee\curlyvee} \sqcup Y^{\curlyvee\curlyvee})^{\curlyvee\curlyvee} = (X \sqcup Y)^{\curlyvee\curlyvee}$.

Proof. From the previous lemma we have $X^{\curlyvee\curlyvee} \sqcup Y^{\curlyvee\curlyvee} \subseteq (X \sqcup Y)^{\curlyvee\curlyvee}$, thus

$$(X^{\curlyvee\curlyvee} \sqcup Y^{\curlyvee\curlyvee})^{\curlyvee\curlyvee} \subseteq (X \sqcup Y)^{\curlyvee\curlyvee}.$$

Further, note that since $X \subseteq X^{\curlyvee\curlyvee}$ and $Y \subseteq Y^{\curlyvee\curlyvee}$ clearly $X \sqcup Y \subseteq X^{\curlyvee\curlyvee} \sqcup Y^{\curlyvee\curlyvee}$ and so

$$(X \sqcup Y)^{\curlyvee\curlyvee} \subseteq (X^{\curlyvee\curlyvee} \sqcup Y^{\curlyvee\curlyvee})^{\curlyvee\curlyvee}.$$

■

Lemma A.2.16.

$$(X^{\curlyvee} \sqcup (Y \cap Z)^{\curlyvee})^{\curlyvee} = (X^{\curlyvee} \sqcup Y^{\curlyvee})^{\curlyvee} \cap (X^{\curlyvee} \sqcup Z^{\curlyvee})^{\curlyvee}.$$

Proof. I prove containment in both directions.

($\subseteq$) Suppose $a \in (X^{\curlyvee} \sqcup (Y \cap Z)^{\curlyvee})^{\curlyvee}$. Note that (via Lemma A.2.12) $a \in (X^{\curlyvee} \sqcup (Y^{\curlyvee} \cup Z^{\curlyvee})^{\curlyvee\curlyvee})^{\curlyvee}$. Next let $b \in X^{\curlyvee}$ and let $c \in Y^{\curlyvee}$ or $c \in Z^{\curlyvee}$. Clearly $c \in (Y^{\curlyvee} \cup Z^{\curlyvee})^{\curlyvee\curlyvee}$. It follows that $a \sqcup b \sqcup c \in \mathbb{I}$. Since we left it open whether $c \in^{\curlyvee}$ or $c \in Z^{\curlyvee}$ it follows that $a \in (X^{\curlyvee} \sqcup Y^{\curlyvee})^{\curlyvee}$ and $a \in (X^{\curlyvee} \sqcup Z^{\curlyvee})^{\curlyvee}$. Thus $a \in (X^{\curlyvee} \sqcup Y^{\curlyvee})^{\curlyvee} \cap (X^{\curlyvee} \sqcup Z^{\curlyvee})^{\curlyvee}$.

($\supseteq$) We reason as follows. First note that

$$X^{\curlyvee} \sqcup (Y^{\curlyvee} \cup Z^{\curlyvee}) = (X^{\curlyvee} \sqcup Y^{\curlyvee}) \cup (X^{\curlyvee} \sqcup Z^{\curlyvee}).$$

Next, it follows that

$$\begin{aligned} X^{\curlyvee\curlyvee\curlyvee} \sqcup (Y^{\curlyvee} \cup Z^{\curlyvee})^{\curlyvee\curlyvee} &= X^{\curlyvee} \sqcup (Y^{\curlyvee} \cup Z^{\curlyvee})^{\curlyvee\curlyvee} \\ &= X^{\curlyvee} \sqcup (Y \cap Z)^{\curlyvee} && \text{(Lemma A.2.12)} \\ &\subseteq ((X^{\curlyvee} \sqcup Y^{\curlyvee}) \cup (X^{\curlyvee} \sqcup Z^{\curlyvee}))^{\curlyvee\curlyvee}. && \text{(Lemma A.2.14)} \end{aligned}$$

Note next that in general $(X^{\curlyvee} \sqcup Y^{\curlyvee}) \subseteq (X^{\curlyvee} \sqcup Y^{\curlyvee})^{\curlyvee\curlyvee}$ and $(X^{\curlyvee} \sqcup Z^{\curlyvee}) \subseteq (X^{\curlyvee} \sqcup Z^{\curlyvee})^{\curlyvee\curlyvee}$. Thus:

$$\begin{aligned} ((X^{\curlyvee} \sqcup Y^{\curlyvee}) \cup (X^{\curlyvee} \sqcup Z^{\curlyvee}))^{\curlyvee\curlyvee} &\subseteq ((X^{\curlyvee} \sqcup Y^{\curlyvee})^{\curlyvee\curlyvee} \cup (X^{\curlyvee} \sqcup Z^{\curlyvee})^{\curlyvee\curlyvee})^{\curlyvee\curlyvee} \\ &= ((X^{\curlyvee} \sqcup Y^{\curlyvee})^{\curlyvee\curlyvee\curlyvee} \cap (X^{\curlyvee} \sqcup Z^{\curlyvee})^{\curlyvee\curlyvee\curlyvee})^{\curlyvee\curlyvee\curlyvee} && \text{(Lemma A.2.12)} \\ &= ((X^{\curlyvee} \sqcup Y^{\curlyvee})^{\curlyvee} \cap (X^{\curlyvee} \sqcup Z^{\curlyvee})^{\curlyvee})^{\curlyvee}. \end{aligned}$$

The result so far is thus:

$$X^{\curlyvee} \sqcup (Y \cap Z)^{\curlyvee} \subseteq ((X^{\curlyvee} \sqcup Y^{\curlyvee})^{\curlyvee} \cap (X^{\curlyvee} \sqcup Z^{\curlyvee})^{\curlyvee})^{\curlyvee}.$$

Corollary A.2.7 thus yields

$$((X^{\curlyvee} \sqcup Y^{\curlyvee})^{\curlyvee} \cap (X^{\curlyvee} \sqcup Z^{\curlyvee})^{\curlyvee}) = ((X^{\curlyvee} \sqcup Y^{\curlyvee})^{\curlyvee} \cap (X^{\curlyvee} \sqcup Z^{\curlyvee})^{\curlyvee})^{\curlyvee\curlyvee} \subseteq (X^{\curlyvee} \sqcup (Y \cap Z)^{\curlyvee})^{\curlyvee}.$$

■

Several interesting facts follow:

Corollary A.2.17.

$$\begin{aligned} [\![A\&(B \vee C)]\!]^P &= [\![(A\&B) \vee (A\&C)]\!]^C \\ [\![A \vee (B\&C)]\!]^C &= [\![(A \vee B)\&(A \vee C)]\!]^C \\ [\![(A\&B) \to C]\!]^P &= [\![(A \to C) \vee (B \to C)]\!]^P \\ [\![(A \vee B) \to C]\!]^C &= [\![(A \to B)\&(A \to C)]\!]^C \end{aligned}$$

Definition A.2.18 (Semantic Entailment)**.** We say that A semantically entails B relative to a model $\mathcal{M}$ if closure of the fuission of A (as premise) and B (as conclusion) consists of only good implications:

$$A \vDash_{\mathcal{M}} B \quad \text{iff}_{df.} \quad ((\llbracket A \rrbracket^P)^\curlyvee \sqcup (\llbracket B \rrbracket^C)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathbb{I}_{\mathcal{M}}.$$

We say that A semantically entails B if $A \vDash_{\mathcal{M}} B$ on all models $\mathcal{M}$.
NB: If A and B are sets of sentences then we read $A \vdash B$ as $\bigwedge A \vdash \bigvee B$, i.e. the conjunction of the elements of A and the disjunction of the elements of B.

In order to simplify matter, it is worth noting that the following holds in the semantics.

Proposition A.2.19. Our semantic clause for '&' in the premises and '∨' in the conclusion is associative. In particular:

$$\llbracket (A\&B)\&C \rrbracket^P = \llbracket A\&(B\&C) \rrbracket^P = \llbracket A\&B\&C \rrbracket^P.$$

Put simpler:

$$(((X^\curlyvee \sqcup Y^\curlyvee)^\curlyvee)^\curlyvee \sqcup Z^\curlyvee)^\curlyvee = (X^\curlyvee \sqcup ((Y^\curlyvee \sqcup Z^\curlyvee)^\curlyvee)^\curlyvee)^\curlyvee = (X^\curlyvee \sqcup Y^\curlyvee \sqcup Z^\curlyvee)^\curlyvee.$$

Proof. Since '⊔' is commutative it will be sufficient to show:

$$(((X^\curlyvee \sqcup Y^\curlyvee)^\curlyvee)^\curlyvee \sqcup Z^\curlyvee)^\curlyvee = (X^\curlyvee \sqcup Y^\curlyvee \sqcup Z^\curlyvee)^\curlyvee.$$

First, we note that $X^\curlyvee \sqcup Y^\curlyvee \subseteq ((X^\curlyvee \sqcup Y^\curlyvee)^\curlyvee)^\curlyvee$. It follows that:

$$X^\curlyvee \sqcup Y^\curlyvee \sqcup Z^\curlyvee \subseteq ((X^\curlyvee \sqcup Y^\curlyvee)^\curlyvee)^\curlyvee \sqcup Z^\curlyvee$$

From Corollary A.2.7, we therefore have that:

$$(((X^\curlyvee \sqcup Y^\curlyvee)^\curlyvee)^\curlyvee \sqcup Z^\curlyvee)^\curlyvee \subseteq (X^\curlyvee \sqcup Y^\curlyvee \sqcup Z^\curlyvee)^\curlyvee.$$

So we need only establish the converse.

Let $a \in (X^\curlyvee \sqcup Y^\curlyvee \sqcup Z^\curlyvee)^\curlyvee$. Next let $x \in X^\curlyvee$, $y \in Y^\curlyvee$ and $z \in Z^\curlyvee$. Clearly $a \sqcup x \sqcup y \sqcup z \in \mathbb{I}$. Since these were all arbitrarily chosen it follows straightaway that $a \sqcup z \in (X^\curlyvee \sqcup Y^\curlyvee)^\curlyvee$. Now suppose $d \in ((X^\curlyvee \sqcup Y^\curlyvee)^\curlyvee)^\curlyvee$. It follows that $a \sqcup z \sqcup d \in \mathbb{I}$. But we may re-arrange this as $d \sqcup z \sqcup a \in \mathbb{I}$, thus

$$a \in (((X^\curlyvee \sqcup Y^\curlyvee)^\curlyvee)^\curlyvee \sqcup Z^\curlyvee)^\curlyvee,$$

and so we are done. ∎

The following is immediate:

Corollary A.2.20.

$$\gamma_1, \ldots, \gamma_n \vDash_{\mathcal{M}} \theta_1, \ldots, \theta_m \Leftrightarrow ((\llbracket\gamma_1\rrbracket^P)^\curlyvee \sqcup \cdots \sqcup (\llbracket\gamma_n\rrbracket^P)^\curlyvee \sqcup (\llbracket\theta_1\rrbracket^C)^\curlyvee \sqcup \cdots \sqcup (\llbracket\theta_m\rrbracket^C)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathcal{I}_{\mathcal{M}}.$$

Unfortunately, due to how I've set things up, no entailments will hold across all models. Here is a short proof.

Proof. Suppose $A \vDash B$. For simplicity, suppose both A and B are atomic. This means that

$$(\langle\{A\}, \emptyset\rangle^{\curlyvee\curlyvee\curlyvee\curlyvee} \sqcup \langle\emptyset, \{B\}\rangle^{\curlyvee\curlyvee\curlyvee\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathbb{I}_{\mathcal{M}},$$

for all models $\mathcal{M}$. We may simplify this as:

$$(\langle\{A\}, \emptyset\rangle^{\curlyvee\curlyvee} \sqcup \langle\emptyset, \{B\}\rangle^{\curlyvee\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathbb{I}_{\mathcal{M}},$$

Since $\langle\{A\}, \emptyset\rangle \in \langle\{A\}, \emptyset\rangle^{\curlyvee\curlyvee}$ and $\langle\{A\}, \emptyset\rangle \in \langle\{B\}, \emptyset\rangle^{\curlyvee\curlyvee}$, it follows that $\langle\{A\}, \{B\}\rangle \in (\langle\{A\}, \emptyset\rangle^{\curlyvee\curlyvee} \sqcup \langle\emptyset, \{B\}\rangle^{\curlyvee\curlyvee})^{\curlyvee\curlyvee}$. Thus $\langle\{A\}, \{B\}\rangle \in \mathbb{I}_{\mathcal{M}}$ for all models $\mathcal{M}$. But clearly nothing we have said forces this result. We may easily construct a model in which $\langle\{A\}, \{B\}\rangle \notin \mathbb{I}$. ■

We therefore require a way of restricting the class of models. To do this I (re-)introduce the notion of a base consequence relation. The choice to restrict our attention to atomic consequence relations is (in a certain sense) arbitrary. We could have just as easily restricted out attention to any kind of consequence relation. The choice to do so in this fashion is designed to complement the proof theory as well as the philosophical account given in the first part of this document.

Definition A.2.21 (Base Consequence Relation)**.** A **base consequence relation** is a subset of $\mathbf{P}$ that consists of only atoms. B is a base consequence relation iff $B \subseteq \mathbf{P}$ and $B \cap \mathcal{P}(\mathcal{L}_0)^2 = B$.

We say that a model $\mathcal{M} = \langle \mathbf{P}, \mathbb{I}, \llbracket\cdot\rrbracket\rangle$ is **fit for** a base consequence relation B iff

$$\forall\langle\Delta, \Lambda\rangle \in B(\Delta \vDash_{\mathcal{M}} \Lambda).$$

We say that Γ semantically entails Θ **relative to** B iff $\Gamma \vDash_{\mathcal{M}} \Theta$ for all models $\mathcal{M}$ fit for B.

A.2.3 Soundness and Completeness

I now prove that the semantics is sound and complete with respect to the sequent calculus. To do so we'll need to enrich how we've formulated things there. I therefore introduce the following notation:

$$\Gamma \vdash_B \Theta,$$

To mean that Γ implies Θ **relative to base consequence relation** B. I shall prove that the proof theory is sound and complete in the sense that (for arbitrary Γ, Θ, and B):

$$\Gamma \vdash_B \Theta \Leftrightarrow \Gamma \vDash_B \Theta.$$

Theorem A.2.22 (Soundness)**.** *The sequent calculus is sound:*

$$\Gamma \vdash_B \Theta \Rightarrow \Gamma \vDash_B \Theta.$$

Proof. Let B be arbitrary. I show that if $\Gamma \vdash_B \Theta$ then $\Gamma \vDash_B \Theta$ by induction on proof height.

In the base case $\Gamma \vdash_B \Theta$ is atomic (i.e. $\Gamma, \Theta \subseteq \mathcal{L}_0$). Since we have restricted our attention to models *fit for* B this case it immediate. That is, $\Gamma \vDash_B \Theta$.

Next suppose that our result holds for proof trees of height strictly less than n and now suppose that $\Gamma \vdash_B \Theta$ comes as the last step of a proof tree of height n. It may come from any of the rules of our sequent calculus (L$\rightarrow$, R$\rightarrow$, L&, R&, L$\vee$, R$\vee$, L$\neg$, R$\neg$).

If the result comes from a rule with a single-top sequent (i.e. R$\rightarrow$, L&, R$\vee$, L$\neg$, or R$\neg$), the result is more or less immediate.

Let us enumerate Γ as $\gamma_1, \ldots, \gamma_n$ and Θ as $\theta_1, \ldots, \theta_n$. Our hypothesis can be restated as (by Corollary A.2.20):

$$((\llbracket\gamma_1\rrbracket^P)^\curlyvee \sqcup \cdots \sqcup (\llbracket\gamma_n\rrbracket^P)^\curlyvee \sqcup (\llbracket\theta_1\rrbracket^C)^\curlyvee \sqcup \cdots \sqcup (\llbracket\theta_m\rrbracket^C)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathcal{I}_\mathcal{M}.$$

First I note the following equivalences:

$$((\llbracket\gamma_i\rrbracket^P)^\curlyvee(\llbracket\theta_j\rrbracket^C)^\curlyvee)^\curlyvee = \llbracket\gamma_i \to \theta_j\rrbracket^C$$

$$((\llbracket\gamma_i\rrbracket^P)^\curlyvee(\llbracket\theta_j\rrbracket^P)^\curlyvee)^\curlyvee = \llbracket\gamma_i \& \gamma_j\rrbracket^P$$

$$((\llbracket\theta_i\rrbracket^C)^\curlyvee(\llbracket\theta_j\rrbracket^C)^\curlyvee)^\curlyvee = \llbracket\theta_i \vee \theta_j\rrbracket^C$$

$$\llbracket\theta_i\rrbracket^C = \llbracket\neg\theta_i\rrbracket^P$$

$$\llbracket\gamma_i\rrbracket^P = \llbracket\neg\gamma_i\rrbracket^C$$

The following is therefore immediate from our inductive hypothesis:

$$((\llbracket\gamma_2\rrbracket^P)^\curlyvee \sqcup \cdots \sqcup (\llbracket\gamma_n\rrbracket^P)^\curlyvee \sqcup (\llbracket\gamma_1 \to \theta_1\rrbracket^C)^\curlyvee \sqcup \llbracket\theta_2\rrbracket^C)^\curlyvee \sqcup \cdots \sqcup (\llbracket\theta_m\rrbracket^C)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathcal{I}_\mathcal{M}$$

$$((\llbracket\gamma_1 \& \gamma_2\rrbracket^P)^\curlyvee \sqcup (\llbracket\gamma_3\rrbracket^P)^\curlyvee \sqcup \cdots \sqcup (\llbracket\gamma_n\rrbracket^P)^\curlyvee \sqcup (\llbracket\theta_1\rrbracket^C)^\curlyvee \sqcup \cdots \sqcup (\llbracket\theta_m\rrbracket^C)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathcal{I}_\mathcal{M}$$

$$((\llbracket\gamma_1\rrbracket^P)^\curlyvee \sqcup \cdots \sqcup (\llbracket\gamma_n\rrbracket^P)^\curlyvee \sqcup (\llbracket\theta_1 \vee \theta_2\rrbracket^C)^\curlyvee \sqcup (\llbracket\theta_3\rrbracket^C)^\curlyvee \sqcup \cdots \sqcup (\llbracket\theta_m\rrbracket^C)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathcal{I}_\mathcal{M}$$

$$((\llbracket\neg\theta_1\rrbracket^P)^\curlyvee \sqcup (\llbracket\gamma_1\rrbracket^P)^\curlyvee \sqcup \cdots \sqcup (\llbracket\gamma_n\rrbracket^P)^\curlyvee \sqcup (\llbracket\theta_2\rrbracket^C)^\curlyvee \sqcup \cdots \sqcup (\llbracket\theta_m\rrbracket^C)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathcal{I}_\mathcal{M}$$

$$((\llbracket\gamma_2\rrbracket^P)^\curlyvee \sqcup \cdots \sqcup (\llbracket\gamma_n\rrbracket^P)^\curlyvee \sqcup (\llbracket\neg\gamma_i\rrbracket^C)^\curlyvee \sqcup (\llbracket\theta_1\rrbracket^C)^\curlyvee \sqcup \cdots \sqcup (\llbracket\theta_m\rrbracket^C)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathcal{I}_\mathcal{M}$$

Thus it follows that

$$\Gamma \vDash_B \Theta,$$

whenever $\Gamma \vdash_B \Theta$ is derivable in proof trees of height n whose last step contains only one-top sequent.

We must show the same when $\Gamma \vdash_B \Theta$ comes via one of our rules with two-top sequents (i.e. L$\to$, R&, and L$\vee$). The key here consists in rewriting our semantic entailment clause as a single set-intersection (that we can do this was established in Lemma A.2.16). I nonetheless explain in detail. The last step in our proof trees is thus:

$$B \to A, \gamma_1, \ldots, \gamma_n \vdash_B \theta_1, \ldots, \theta_m$$

$$\gamma_1, \ldots, \gamma_n \vdash_B \theta_1, \ldots, \theta_m, A \& B$$

$$A \vee B, \gamma_1, \ldots, \gamma_n \vdash_B \theta_1, \ldots, \theta_m$$

The top sequents in these cases shall be

$$A, \gamma_1, \ldots, \gamma_n \vdash_B \theta_1, \ldots, \theta_m \quad \text{and} \quad \gamma_1, \ldots, \gamma_n \vdash_B \theta_1, \ldots, \theta_m, B \tag{L$\rightarrow$}$$

$$\gamma_1, \ldots, \gamma_n \vdash_B \theta_1, \ldots, \theta_m, A \quad \text{and} \quad \gamma_1, \ldots, \gamma_n \vdash_B \theta_1, \ldots, \theta_m, B \tag{R\&}$$

$$A, \gamma_1, \ldots, \gamma_n \vdash_B \theta_1, \ldots, \theta_m \quad \text{and} \quad B, \gamma_1, \ldots, \gamma_n \vdash_B \theta_1, \ldots, \theta_m \tag{L$\vee$}$$

I'll here simply write $[\![\&\Gamma]\!]^P$ and $[\![\bigvee\Theta]\!]^C$ for brevity. Our inductive hypothesis guarantees the following:

$$(([\![\&\Gamma]\!]^P)^\curlyvee \sqcup ([\![\bigvee\Theta]\!]^C)^\curlyvee \sqcup ([\![A]\!]^P)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathbb{I}_\mathcal{M} \quad \text{and} \quad (([\![\&\Gamma]\!]^P)^\curlyvee \sqcup ([\![\bigvee\Theta]\!]^C)^\curlyvee \sqcup ([\![B]\!]^C)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathbb{I}_\mathcal{M} \tag{L$\rightarrow$}$$

$$(([\![\&\Gamma]\!]^P)^\curlyvee \sqcup ([\![\bigvee\Theta]\!]^C)^\curlyvee \sqcup ([\![A]\!]^C)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathbb{I}_\mathcal{M} \quad \text{and} \quad (([\![\&\Gamma]\!]^P)^\curlyvee \sqcup ([\![\bigvee\Theta]\!]^C)^\curlyvee \sqcup ([\![B]\!]^C)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathbb{I}_\mathcal{M} \tag{R\&}$$

$$(([\![\&\Gamma]\!]^P)^\curlyvee \sqcup ([\![\bigvee\Theta]\!]^C)^\curlyvee \sqcup ([\![A]\!]^P)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathbb{I}_\mathcal{M} \quad \text{and} \quad (([\![\&\Gamma]\!]^P)^\curlyvee \sqcup ([\![\bigvee\Theta]\!]^C)^\curlyvee \sqcup ([\![B]\!]^P)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathbb{I}_\mathcal{M} \tag{L$\vee$}$$

Next, recall that because '$\curlyvee\curlyvee$' functions as a closure operation, it follows that if $X \subseteq Y$ that $X^{\curlyvee\curlyvee} \subseteq Y^{\curlyvee\curlyvee}$. I also observe in general that $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$. Hence, we have the following:

$$[\Big(([\![\&\Gamma]\!]^P)^\curlyvee \sqcup ([\![\bigvee\Theta]\!]^C)^\curlyvee \sqcup ([\![A]\!]^P)^\curlyvee\Big) \cap \Big(([\![\&\Gamma]\!]^P)^\curlyvee \sqcup ([\![\bigvee\Theta]\!]^C)^\curlyvee \sqcup ([\![B]\!]^C)^\curlyvee\Big)]^{\curlyvee\curlyvee} \subseteq \mathbb{I}_\mathcal{M} \tag{L$\rightarrow$}$$

$$[\Big(([\![\&\Gamma]\!]^P)^\curlyvee \sqcup ([\![\bigvee\Theta]\!]^C)^\curlyvee \sqcup ([\![A]\!]^C)^\curlyvee\Big) \cap \Big(([\![\&\Gamma]\!]^P)^\curlyvee \sqcup ([\![\bigvee\Theta]\!]^C)^\curlyvee \sqcup ([\![B]\!]^C)^\curlyvee\Big)]^{\curlyvee\curlyvee} \subseteq \mathbb{I}_\mathcal{M} \tag{R\&}$$

$$[\Big(([\![\&\Gamma]\!]^P)^\curlyvee \sqcup ([\![\bigvee\Theta]\!]^C)^\curlyvee \sqcup ([\![A]\!]^P)^\curlyvee\Big) \cap \Big(([\![\&\Gamma]\!]^P)^\curlyvee \sqcup ([\![\bigvee\Theta]\!]^C)^\curlyvee \sqcup ([\![B]\!]^P)^\curlyvee\Big)]^{\curlyvee\curlyvee} \subseteq \mathbb{I}_\mathcal{M} \tag{L$\vee$}$$

Via Lemma A.2.16 we may rewrite the above as:

$$(([\![\&\Gamma]\!]^P)^\curlyvee \sqcup ([\![\bigvee\Theta]\!]^C)^\curlyvee \sqcup ([\![A]\!]^P \cap [\![B]\!]^C)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathbb{I}_\mathcal{M} \tag{L$\rightarrow$}$$

$$(([\![\&\Gamma]\!]^P)^\curlyvee \sqcup ([\![\bigvee\Theta]\!]^C)^\curlyvee \sqcup ([\![A]\!]^C \cap [\![B]\!]^C)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathbb{I}_\mathcal{M} \tag{R\&}$$

$$(([\![\&\Gamma]\!]^P)^\curlyvee \sqcup ([\![\bigvee\Theta]\!]^C)^\curlyvee \sqcup ([\![A]\!]^P \cap [\![B]\!]^P)^\curlyvee)^{\curlyvee\curlyvee} \subseteq \mathbb{I}_\mathcal{M} \tag{L$\vee$}$$

Next I note the following equivalences:

$$[\![A]\!]^P \cap [\![B]\!]^C = [\![B \to A]\!]^P$$
$$[\![A]\!]^C \cap [\![B]\!]^C = [\![A\&B]\!]^C$$
$$[\![A]\!]^P \cap [\![B]\!]^P = [\![A \vee B]\!]^P$$

Thus, it follows immediately that

$$B \to A, \gamma_1, \dots, \gamma_n \vDash_B \theta_1, \dots, \theta_m$$
$$\gamma_1, \dots, \gamma_n \vDash_B \theta_1, \dots, \theta_m, A\&B$$
$$A \vee B, \gamma_1, \dots, \gamma_n \vDash_B \theta_1, \dots, \theta_m$$

It therefore follows in all cases that:

$$\Gamma \vdash_B \Theta \Rightarrow \Gamma \vDash_B \Theta.$$

■

Theorem A.2.23 (Completeness)**.** *The sequent calculus is complete:*

$$\Gamma \vDash_B \Theta \Rightarrow \Gamma \vdash_B \Theta.$$

Proof. I show the contrapositive by constructing canonical models on which $\Gamma \vdash \Theta$ iff $\Gamma \vDash_{\mathcal{M}} \Theta$. Hence I wish to show that:

$$\Gamma \nvdash_B \Theta \Rightarrow \Gamma \nvDash_B \Theta.$$

Thus suppose that $\Gamma \nvdash_B \Theta$. I construct a model $\mathcal{M} = \langle \mathbf{P}, \mathbb{I}, [\![\cdot]\!] \rangle$ fit for B on which $\Gamma \nvDash_{\mathcal{M}} \Theta$.

Setting up the canonical models is fairly straightforward. We simply let $\mathbb{I}_{\mathcal{M}} = \vdash_B$, i.e.:[14]

$$\mathbb{I}_{\mathcal{M}} = \{\langle \Gamma, \Delta \rangle | \Gamma \vdash_B \Theta\}.$$

[14] I have chosen this because I consider it a more elegant solution. It would be equally sufficient, however, simply to stipulate that $\mathbb{I}_{\mathcal{M}} = \mid\!\sim_B \cap \mathcal{P}(\mathcal{L}_0)$ (i.e. contains all and only the atomic sequents). The crucial difference, however, concerns setting up the interpretation function.

Next, I prove that the interpretation function meets the following constraint:

$$[\![A]\!] = \langle \langle \{A\}, \emptyset \rangle^{\curlyvee}, \langle \emptyset, \{A\} \rangle^{\curlyvee} \rangle.$$

I previously defined interpretation functions such that the above always hold of *atomic* sentences, but it need not in general hold for all sentences (in all models). Really, what I'll be proving then is that in canonical models, the interpretation function works as follows:

$$[\![A]\!] = \langle \{\langle \Gamma, \Theta \rangle | A, \Gamma \vdash_B \Theta\}, \{\langle \Gamma, \Theta \rangle | \Gamma \vdash_B \Theta, A\} \rangle$$

Note that it should be immediately obvious that these two equations are equivalent given the choice of $\mathbb{I}_{\mathcal{M}}$. Once I've shown this it will follow that all and only good implications turn out to be semantic entailments.

In order to keep things straight, I will therefore introduce new notation for our intended interpretation function: $(\!|\cdot|\!)$. Thus let

$$(\!|A|\!) = \langle \{\langle \Gamma, \Theta \rangle | A, \Gamma \vdash_B \Theta\}, \{\langle \Gamma, \Theta \rangle | \Gamma \vdash_B \Theta, A\} \rangle = \langle \langle \{A\}, \emptyset \rangle^{\curlyvee}, \langle \emptyset, \{A\} \rangle^{\curlyvee} \rangle.$$

I now show that $(\!|A|\!) = [\![A]\!]$ by induction on logical complexity.

For the base case suppose A is atomic, then we have our result. Since $\mathbb{I}_{\mathcal{M}} = \vdash_B$ it follows that

$$\begin{aligned} [\![A]\!] &= \langle \langle \{A\}, \emptyset \rangle^{\curlyvee}, \langle \emptyset, \{A\} \rangle^{\curlyvee} \rangle \\ &= \langle \{\langle \Gamma, \Theta \rangle | A, \Gamma \vdash_B \Theta\}, \{\langle \Gamma, \Theta \rangle | \Gamma \vdash_B \Theta, A\} \rangle \\ &= (\!|A|\!). \end{aligned}$$

Thus suppose our result holds for sentences of logical complexity strictly less than n and now suppose that A has complexity of n. A is either a conditional, conjunction, disjunction, or negation. To make navigation easier I will use large headings for each of these cases. In each case I shall show $[\![A]\!] = (\!|A|\!)$ by showing containment in both directions for each member of the ordered pair. Note that $(\!|A|\!)^P$ and $(\!|A|\!)^C$ are used in the same way as $[\![A]\!]^P$ and $[\![A]\!]^C$.

Case 1: Conditional

Suppose A is a conditional, then it is of the form $B \to C$. I treat $[\![B \to C]\!]^P$ and $[\![B \to C]\!]^C$ separately. I am quite thorough in handling the conditional which can make following the "larger argument" more tedious. Consulting the corresponding proofs for the conjunction and disjunction may prove helpful.

I first show $[\![B \to C]\!]^P = (\!|B \to C|\!)^P$I. By our connective definition we have:

$$[\![B \to C]\!]^P = [\![B]\!]^C \cap [\![C]\!]^P.$$

By our inductive hypothesis, therefore:

$$[\![B]\!]^C \cap [\![C]\!]^P = \langle \emptyset, \{B\}\rangle^{\curlyvee} \cap \langle \{C\}, \emptyset\rangle^{\curlyvee}.$$

This is of course equivalent to

$$\langle \emptyset, \{B\}\rangle^{\curlyvee} \cap \langle \{C\}, \emptyset\rangle^{\curlyvee} = \{\langle \Gamma, \Theta\rangle | \Gamma \vdash_B \Theta, B \quad \text{and} \quad C, \Gamma \vdash_B \Theta\}.$$

In the sequent calculus our rule for L$\to$ has it that $\Gamma \vdash_B \Theta, B$ and $C, \Gamma \vdash_B \Theta$ iff $B \to C, \Gamma \vdash_B \Theta$. Thus we may rewrite the above:

$$\{\langle \Gamma, \Theta\rangle | \Gamma \vdash_B \Theta, B \quad \text{and} \quad C, \Gamma \vdash_B \Theta\} = \{\langle \Gamma, \Theta\rangle | B \to C, \Gamma \vdash_B \Theta\} \quad = \langle \{B \to C\}, \emptyset\rangle^{\curlyvee} = (\!|B \to C|\!)^P.$$

Thus

$$[\![B \to C]\!]^P = (\!|B \to C|\!)^P.$$

Next I show $[\![B \to C]\!]^C = (\!|B \to C|\!)^C$. From our semantic definition, we have

$$[\![B \to C]\!]^C = (([\![B]\!]^P)^{\curlyvee} \sqcup ([\![C]\!]^C)^{\curlyvee})^{\curlyvee}.$$

From our inductive hypothesis, this is therefore

$$(([\![B]\!]^P)^{\curlyvee} \sqcup ([\![C]\!]^C)^{\curlyvee})^{\curlyvee} = (\langle \{B\}, \emptyset\rangle^{\curlyvee\curlyvee} \sqcup \langle \emptyset, \{C\}\rangle^{\curlyvee\curlyvee})^{\curlyvee}.$$

($\subseteq$) Now, obviously $\langle \{B\}, \emptyset\rangle \in \langle \{B\}, \emptyset\rangle^{\curlyvee\curlyvee}$ and $C \in \langle \emptyset, \{C\}\rangle^{\curlyvee\curlyvee}$. Thus $\langle \{B\}, \{C\}\rangle \in \langle \{B\}, \emptyset\rangle^{\curlyvee\curlyvee} \sqcup \langle \emptyset, \{C\}\rangle^{\curlyvee\curlyvee}$.

Now, let $\langle \Delta, \Lambda\rangle \in [\![B \to C]\!]^C$ be arbitrary. Clearly $\langle \Delta, \Lambda\rangle \sqcup \langle \{B\}, \{C\}\rangle \in \mathbb{I}_{\mathcal{M}}$. Thus, $\langle \Delta, \Lambda\rangle \in \{\langle \Gamma, \Theta\rangle | \Gamma, B \vdash_B C, \Theta\}$. In the sequent calculus our rule for R$\to$ has it that if

$\Gamma, B \vdash_B C, \Theta$, then $\Gamma \vdash_B \Theta, A \to B$. Thus, $\langle \Delta, \Lambda \rangle \in \{\langle \Gamma, \Theta \rangle | \Gamma \vdash_B \Theta, B \to C\}$, i.e. $\langle \Delta, \Lambda \rangle \in \langle \emptyset, \{B \to C\} \rangle^{\curlyvee}$ and so in $(\!| B \to C |\!)^C$. It follows that

$$\llbracket B \to C \rrbracket^C \subseteq (\!| B \to C |\!)^C.$$

($\supseteq$) First we note that:

$$\begin{aligned}(\!| B \to C |\!)^C &= \langle \emptyset, \{B \to C\} \rangle^{\curlyvee} \\ &= \{\langle \Gamma, \Theta \rangle | \Gamma \vdash_B \Theta, B \to C\}.\end{aligned}$$

Since our rule for R$\to$, we have $\Gamma \vdash_B \Theta, B \to C$ iff $\Gamma, B \vdash_B C, \Theta$, thus

$$\begin{aligned}\{\langle \Gamma, \Theta \rangle | \Gamma \vdash_B \Theta, B \to C\} &= \{\langle \Gamma, \Theta \rangle | \Gamma, B \vdash_B C, \Theta\} \\ &= \langle \{B\}, \{C\} \rangle^{\curlyvee} \\ &= \langle \{B\}, \{C\} \rangle^{\curlyvee\curlyvee\curlyvee} \\ &= (\langle \{B\}, \emptyset \rangle \sqcup \langle \emptyset, \{C\} \rangle)^{\curlyvee\curlyvee\curlyvee}.\end{aligned}$$

Now, we have that via Lemma A.2.14 that $\langle \{B\}, \emptyset \rangle^{\curlyvee\curlyvee} \sqcup \langle \emptyset, \{C\} \rangle^{\curlyvee\curlyvee} \subseteq (\langle \{B\}, \emptyset \rangle \sqcup \langle \emptyset, \{C\} \rangle)^{\curlyvee\curlyvee}$. Hence via Corollary A.2.7 we have that

$$(\langle \{B\}, \emptyset \rangle \sqcup \langle \emptyset, \{C\} \rangle)^{\curlyvee\curlyvee\curlyvee} \subseteq (\langle \{B\}, \emptyset \rangle^{\curlyvee\curlyvee} \sqcup \langle \emptyset, \{C\} \rangle^{\curlyvee\curlyvee})^{\curlyvee}.$$

From our inductive hypothesis we have

$$\begin{aligned}(\langle \{B\}, \emptyset \rangle^{\curlyvee\curlyvee} \sqcup \langle \emptyset, \{C\} \rangle^{\curlyvee\curlyvee})^{\curlyvee} &= ((\llbracket B \rrbracket^P)^{\curlyvee} \sqcup (\llbracket C \rrbracket^C)^{\curlyvee})^{\curlyvee} \\ &= \llbracket B \to C \rrbracket^C.\end{aligned}$$

Thus $(\!| B \to C |\!)^C \subseteq \llbracket B \to C \rrbracket^C$.

Since we have containment in both directions we therefore have $\llbracket B \to C \rrbracket^C = (\!| B \to C |\!)^C$ and thus

$$\llbracket B \to C \rrbracket = (\!| B \to C |\!).$$

Case 2: Conjunction

Suppose A is a conjunction, then it is of the form $B\&C$. I treat $[\![B\&C]\!]^P$ and $[\![B\&C]\!]^C$ separately. Since both cases are analogous to the proofs concerning the conditional I will be much briefer here.

To start we wish to show $[\![B\&C]\!]^P = (\!|B\&C|\!)^P$. The proof is similar to establishing the analogous result for the conclusory role of the conditional. In that case, the proof can be reduced to establishing one key step, which I mark here using (***):

$$\begin{aligned}
[\![B\&C]\!]^P &= (([\![B]\!]^P)^\curlyvee \sqcup ([\![C]\!]^P)^\curlyvee)^\curlyvee \\
&= (\langle\{B\},\emptyset\rangle^{\curlyvee\curlyvee} \sqcup \langle\{C\},\emptyset\rangle^{\curlyvee\curlyvee})^\curlyvee && \text{(Inductive Hypothesis)} \\
&\stackrel{?}{=} (\langle\{B\},\emptyset\rangle \sqcup \langle\{C\},\emptyset\rangle)^{\curlyvee\curlyvee\curlyvee} && \text{(***)} \\
&= \langle\{B,C\},\emptyset\rangle^{\curlyvee\curlyvee\curlyvee} \\
&= \langle\{B,C\},\emptyset\rangle^{\curlyvee} \\
&= \{\langle\Gamma,\Theta\rangle | A, B, \Gamma \vdash_B \Theta\} \\
&= \{\langle\Gamma,\Theta\rangle | A\&B, \Gamma \vdash_B \Theta\} && \text{(L\& Rule)} \\
&= \langle\{B\&C\},\emptyset\rangle^{\curlyvee} \\
&= (\!|B\&C|\!)^P.
\end{aligned}$$

Thus the proof of $[\![B\&C]\!]^P = (\!|B\&C|\!)^P$ reduces to

$$(\langle\{B\},\emptyset\rangle^{\curlyvee\curlyvee} \sqcup \langle\{C\},\emptyset\rangle^{\curlyvee\curlyvee})^\curlyvee \stackrel{?}{=} (\langle\{B\},\emptyset\rangle \sqcup \langle\{C\},\emptyset\rangle)^{\curlyvee\curlyvee\curlyvee}. \qquad \text{(***)}$$

($\subseteq$) Let $\langle\Delta,\Lambda\rangle \in (\langle\{B\},\emptyset\rangle^{\curlyvee\curlyvee} \sqcup \langle\{C\},\emptyset\rangle^{\curlyvee\curlyvee})^\curlyvee$ be arbitrary. Next, observe that $\langle\{B\},\emptyset\rangle \in \langle\{B\},\emptyset\rangle^{\curlyvee\curlyvee}$ and $\langle\{C\},\emptyset\rangle \in \langle\{C\},\emptyset\rangle^{\curlyvee\curlyvee}$ and hence $\langle\{B\},\emptyset\rangle \sqcup \langle\{C\},\emptyset\rangle \in \langle\{B\},\emptyset\rangle^{\curlyvee\curlyvee} \sqcup \langle\{C\},\emptyset\rangle^{\curlyvee\curlyvee}$. It follows that $\langle\{B\},\emptyset\rangle \sqcup \langle\{C\},\emptyset\rangle \sqcup \langle\Delta,\Lambda\rangle \in \mathbb{I}_{\mathcal{M}}$ and thus $\langle\Delta,\Lambda\rangle \in (\langle\{B\},\emptyset\rangle \sqcup \langle\{C\},\emptyset\rangle)^\curlyvee$.

($\supseteq$) We have (via Lemma A.2.14) that

$$\langle\{B\},\emptyset\rangle^{\curlyvee\curlyvee} \sqcup \langle\{C\},\emptyset\rangle^{\curlyvee\curlyvee} \subseteq (\langle\{B\},\emptyset\rangle \sqcup \langle\{C\},\emptyset\rangle)^{\curlyvee\curlyvee}$$

Hence via Corollary A.2.7 we have:

$$(\langle\{B\},\emptyset\rangle \sqcup \langle\{C\},\emptyset\rangle)^{\curlyvee\curlyvee\curlyvee} \subseteq (\langle\{B\},\emptyset\rangle^{\curlyvee\curlyvee} \sqcup \langle\{C\},\emptyset\rangle^{\curlyvee\curlyvee})^\curlyvee.$$

Hence,

$$\llbracket B\&C \rrbracket^P = (\!| B\&C |\!)^P.$$

Next, I show that $\llbracket B\&C \rrbracket^C = (\!| B\&C |\!)^C$. Since this case is very similar to the analogous result for the premissory role of the conditional, I allow myself to be much briefer here. I reason as follows:

$$\begin{aligned}
\llbracket B\&C \rrbracket^C &= \llbracket B \rrbracket^C \cap \llbracket C \rrbracket^C \\
&= \langle \emptyset, \{B\} \rangle^{\curlyvee} \cap \langle \emptyset, \{C\} \rangle^{\curlyvee} && \text{(Inductive Hypothesis)} \\
&= \{\langle \Gamma, \Theta \rangle | \Gamma \vdash_B \Theta, B \quad \text{and} \quad \Gamma \vdash_B \Theta, C\} \\
&= \{\langle \Gamma, \Theta \rangle | \Gamma \vdash_B \Theta, B\&C\} && \text{(R\&)} \\
&= \langle \emptyset, \{B\&C\} \rangle^{\curlyvee} && = (\!| B\&C |\!)^C.
\end{aligned}$$

We therefore have that $\llbracket B\&C \rrbracket^C = (\!| B\&C |\!)^C$, and thus:

$$\llbracket B\&C \rrbracket = (\!| B\&C |\!).$$

Case 3: Disjunction

Suppose A is a disjunction, then it is of the form $B \vee C$. I treat $\llbracket B \vee C \rrbracket^P$ and $\llbracket B \vee C \rrbracket^C$ separately. Since both cases are analogous to the proofs concerning the conditional and conjunction I will be much briefer here.

I first show that $\llbracket B \vee C \rrbracket^P = (\!| B \vee C |\!)^P$. Since this case is very similar to the analogous result for the premissory role of the conditional and conclusory role of the conjunction, I allow myself to be much briefer here. I reason as follows:

$$\begin{aligned}
\llbracket B \vee C \rrbracket^P &= \llbracket B \rrbracket^P \cap \llbracket C \rrbracket^P \\
&= \langle \{B\}, \emptyset \rangle^{\curlyvee} \cap \langle \{C\}, \emptyset \rangle^{\curlyvee} && \text{(Inductive Hypothesis)} \\
&= \{\langle \Gamma, \Theta \rangle | B, \Gamma \vdash_B \Theta \quad \text{and} \quad C, \Gamma \vdash_B \Theta\} \\
&= \{\langle \Gamma, \Theta \rangle | B \vee C, \Gamma \vdash_B \Theta\} && \text{(L}\vee\text{)} \\
&= \langle \{B \vee C\}, \emptyset \rangle^{\curlyvee} && = (\!| B \vee C |\!)^P.
\end{aligned}$$

We therefore have that

$$\llbracket B \vee C \rrbracket^P = (\!| B \vee C |\!)^P.$$

Next I show $[\![B \vee C]\!]^C = (\!|B \vee C|\!)^C$. I allow myself to be briefer here. As with the conjunction, we can mark the crucial step with (***):

$$
\begin{aligned}
[\![B \vee C]\!]^C &= (([\![B]\!]^C)^\curlyvee \sqcup ([\![C]\!]^C)^\curlyvee)^\curlyvee \\
&= (\langle \emptyset, \{B\}\rangle^{\curlyvee\curlyvee} \sqcup \langle \emptyset, \{C\}\rangle^{\curlyvee\curlyvee})^\curlyvee && \text{(Inductive Hypothesis)} \\
&\stackrel{?}{=} (\langle \emptyset, \{B\}\rangle \sqcup \langle \emptyset, \{C\}\rangle)^{\curlyvee\curlyvee\curlyvee} && \text{(***)} \\
&= \langle \emptyset, \{B, C\}\rangle^{\curlyvee\curlyvee\curlyvee} \\
&= \langle \emptyset, \{B, C\}\rangle^{\curlyvee} \\
&= \{\langle \Gamma, \Theta\rangle | \Gamma \vdash_B \Theta, B, C\} \\
&= \{\langle \Gamma, \Theta\rangle | \Gamma \vdash_B \Theta, B, C\} && \text{(R}\vee\text{ Rule)} \\
&= \langle \emptyset, \{B \vee C\}\rangle^{\curlyvee} \\
&= (\!|B \vee C|\!)^C.
\end{aligned}
$$

Thus the proof reduces to

$$(\langle \emptyset, \{B\}\rangle^{\curlyvee\curlyvee} \sqcup \langle \emptyset, \{C\}\rangle^{\curlyvee\curlyvee})^\curlyvee \stackrel{?}{=} (\langle \emptyset, \{B\}\rangle \sqcup \langle \emptyset, \{C\}\rangle)^{\curlyvee\curlyvee\curlyvee}.$$

($\subseteq$) Let $\langle \Delta, \Lambda\rangle \in (\langle \emptyset, \{B\}\rangle^{\curlyvee\curlyvee} \sqcup \langle \emptyset, \{C\}\rangle^{\curlyvee\curlyvee})^\curlyvee$. Clearly $\langle \emptyset, \{B\}\rangle \sqcup \langle \emptyset, \{C\}\rangle \in \langle \emptyset, \{B\}\rangle^{\curlyvee\curlyvee} \sqcup \langle \emptyset, \{C\}\rangle^{\curlyvee\curlyvee}$. Thus $\langle \emptyset, \{B\}\rangle \sqcup \langle \emptyset, \{C\}\rangle \sqcup \langle \Delta, \Lambda\rangle \in \mathbb{I}_{\mathcal{M}}$.

($\supseteq$) We have (via Lemma A.2.14)

$$(\langle \emptyset, \{B\}\rangle^{\curlyvee\curlyvee} \sqcup \langle \emptyset, \{C\}\rangle^{\curlyvee\curlyvee}) \subseteq (\langle \emptyset, \{B\}\rangle \sqcup \langle \emptyset, \{C\}\rangle)^{\curlyvee\curlyvee},$$

and thus via Corollary A.2.7:

$$(\langle \emptyset, \{B\}\rangle \sqcup \langle \emptyset, \{C\}\rangle)^{\curlyvee\curlyvee\curlyvee} \subseteq (\langle \emptyset, \{B\}\rangle^{\curlyvee\curlyvee} \sqcup \langle \emptyset, \{C\}\rangle^{\curlyvee\curlyvee})^\curlyvee.$$

Hence, $[\![B \vee C]\!]^C = (\!|B \vee C|\!)^C$, and thus

$$[\![B \vee C]\!] = (\!|B \vee C|\!).$$

Case 4: Negation

Finally, suppose that A is a negated sentence, i.e. A has the form $\neg B$. We must show $[\![\neg B]\!] = (\!|\neg B|\!)$. I here treat the cases of $[\![B]\!]^P$ and $[\![B]\!]^C$ simultaneously since they are handled analogously and since the general principles of the proof should be familiar from the previous cases. I reason as follows:

$$
\begin{aligned}
[\![\neg B]\!] &= \langle [\![B]\!]^C, [\![B]\!]^P \rangle \\
&= \langle \langle \emptyset, \{B\} \rangle^{\curlyvee}, \langle \{B\}, \emptyset \rangle^{\curlyvee} \rangle && \text{(Inductive Hypothesis)} \\
&= \langle \{ \langle \Gamma, \Theta \rangle | \Gamma \vdash_B \Theta, B \}, \{ \langle \Gamma, \Theta \rangle | B, \Gamma \vdash_B \Theta \} \rangle \\
&= \langle \{ \langle \Gamma, \Theta \rangle | \neg B, \Gamma \vdash_B \Theta \}, \{ \langle \Gamma, \Theta \rangle | \Gamma \vdash_B \Theta, \neg B \} \rangle && \text{(L}\neg\text{/ R}\neg\text{)} \\
&= \langle \langle \{\neg B\}, \emptyset \rangle^{\curlyvee}, \langle \emptyset, \{\neg B\} \rangle^{\curlyvee} \rangle \\
&= (\!|B|\!).
\end{aligned}
$$

Note that the step labeled (L$\neg$/R$\neg$) follows since in our sequent calculus $\Gamma \vdash_B \Theta, B$ iff $\neg B, \Gamma \vdash_B \Theta$ and likewise $B, \Gamma \vdash_B \Theta$ iff $\Gamma \vdash_B \Theta, \neg B$, i.e. our '$\neg$' is involutive.

We therefore have that $[\![\neg B]\!] = (\!|\neg B|\!)$. Since we have shown this in all cases it follows that

$$[\![A]\!] = (\!|A|\!).$$

Main Proof Body

Recall the point of establishing this about our canonical models was to show that $\Gamma \vdash_B \Theta$ iff $\Gamma \vDash_{\mathcal{M}} \Theta$ on such models. I next show this. Since many of the principles of the proof are similar to what has already been shown I allow myself some brevity here. Let us enumerate Γ and Θ as $\gamma_1, \ldots, \gamma_n$ and $\theta_1, \ldots, \theta_m$. I reason as follows

$$
\begin{aligned}
\Gamma \vDash_{\mathcal{M}} \Theta &\Leftrightarrow (([\![\gamma_1]\!]^P)^{\curlyvee} \sqcup \cdots \sqcup ([\![\gamma_n]\!]^P)^{\curlyvee} \sqcup ([\![\theta_1]\!]^C)^{\curlyvee} \sqcup \cdots \sqcup ([\![\theta_m]\!]^C)^{\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathcal{I}_{\mathcal{M}} \\
&\Leftrightarrow (((\!|\gamma_1|\!)^P)^{\curlyvee} \sqcup \cdots \sqcup ((\!|\gamma_n|\!)^P)^{\curlyvee} \sqcup ((\!|\theta_1|\!)^C)^{\curlyvee} \sqcup \cdots \sqcup ((\!|\theta_m|\!)^C)^{\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathcal{I}_{\mathcal{M}} \\
&\Leftrightarrow (\langle \{\gamma_1\}, \emptyset \rangle^{\curlyvee\curlyvee} \sqcup \cdots \sqcup \langle \{\gamma_n\}, \emptyset \rangle^{\curlyvee\curlyvee} \sqcup \langle \{\theta_1\}, \emptyset \rangle^{\curlyvee\curlyvee} \sqcup \cdots \sqcup \langle \{\theta_m\}, \emptyset \rangle^{\curlyvee\curlyvee})^{\curlyvee\curlyvee} \subseteq \mathcal{I}_{\mathcal{M}} \\
&\Leftrightarrow (\langle \{\gamma_1\}, \emptyset \rangle \sqcup \cdots \sqcup \langle \{\gamma_n\}, \emptyset \rangle \sqcup \langle \{\theta_1\}, \emptyset \rangle \sqcup \cdots \sqcup \langle \{\theta_m\}, \emptyset \rangle)^{\curlyvee\curlyvee\curlyvee\curlyvee} \subseteq \mathcal{I}_{\mathcal{M}} \\
&\Leftrightarrow \langle \{\gamma_1, \ldots, \gamma_n\}, \{\theta_1, \ldots, \theta_m\} \rangle^{\curlyvee\curlyvee} \subseteq \mathcal{I}_{\mathcal{M}} \\
&\Leftrightarrow \langle \Gamma, \Theta \rangle^{\curlyvee\curlyvee} \subseteq \mathcal{I}_{\mathcal{M}} \\
&\Leftrightarrow \langle \Gamma, \Theta \rangle \in \mathbb{I}_{\mathcal{M}} && \text{(A.2.8)} \\
&\Leftrightarrow \Gamma \vdash_B \Theta.
\end{aligned}
$$

It follows that

$$\Gamma \vDash_{\mathcal{M}} \Theta \Leftrightarrow \Gamma \vdash_B \Theta.$$

Thus if $\Gamma \nvdash_B \Theta$, then $\Gamma \nvDash_B \Theta$. Hence

$$\Gamma \vDash_B \Theta \Rightarrow \Gamma \vdash_B \Theta.$$

■

From these two theorems we have

$$\Gamma \vdash_B \Theta \Leftrightarrow \Gamma \vDash_B \Theta,$$

that is, the sequent calculus is sound and complete with respect to the inferential role semantics.

A.2.4 Functional Completeness of Connectives

In this section I want to distinguish two senses in which the connectives may be "functionally complete". The first sense is meant to pick out the idea that for any $X^{\curlyvee\curlyvee} \subseteq \mathbf{P}$, we may find a sentence $A \in \mathcal{L}$ such that either $[\![A]\!]^P = X^{\curlyvee\curlyvee}$ or $[\![A]\!]^C = X^{\curlyvee\curlyvee}$. In other words, the connectives are expressive enough that any closed set will play the role of either premises or conclusion for *some* sentence.

The second sense has it that for any $\langle X^{\curlyvee\curlyvee}, Y^{\curlyvee\curlyvee} \rangle \subseteq \mathbf{P}^2$ (i.e. for any *proper inferential role*) there is some sentence that picks out *exactly that role.*

I define the notions of functional completeness. I simply call them "functional completeness 1" (Φ_1) and "functional completeness 2" (Φ_2) for now (lacking a better name).

Some important things to note:

- Functional completeness is a property that obtains between a base consequence relation and a set of connectives. Thus, my connectives $\{\&, \vee, \rightarrow, \neg\}$ are functionally complete in neither sense, but if we restrict B to classical logic: minimal, flat, then $\{\&, \vee, \rightarrow, \neg\}$ *is* functionally complete in the first, but not the second sense.

- This notion will be important in the section following this since Φ_1 turns out to be sufficient for $\Delta_4 \Rightarrow \Delta_3$. Φ_2 is important since it turns out that Φ_2 is sufficient for $\Delta_3 \Rightarrow \Delta_2$.
- Φ_2 isn't necessarily a desirable quality.

A.2.5 Theories

In line with material pursued in the previous appendix, we can also limit models to realize or satisfy theories. I have already detailed in the proof theory how this works and since the model theory is complete we can obviously perform the same sorts of restrictions within it (i.e. we can simply impose constraints on $\vdash_0$, e.g. on models fit for $B = \vdash_0$).

In the case where we impose theories (i.e. sets of sentences of inferential relations) how this will work is obvious. We simply read off from the theory/inferential relation what must be included in B (or what B must be) to ensure that we get the proper restriction on models.

Analogously we can read similar *structural* constraints in the same way. Since the structural constraints I considered in the previous section were simply reducible to constraints on B, we can make similar enrichments here.

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